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JUAN ELIAS *, ROSER HOMS **, AND BERNARD MOURRAIN ***

Abstract. We analyze and present an effective solution to the minimal Gorenstein cover problem: given a local Artin $\mathbf{k}$-algebra $A = \mathbf{k}[\![x_1, \ldots, x_n]\!]/I$, compute an Artin Gorenstein $\mathbf{k}$-algebra $G = \mathbf{k}[\![x_1, \ldots, x_n]\!]/J$ such that $\ell(G) - \ell(A)$ is minimal. We approach the problem by using Macaulay's inverse systems and a modification of the integration method for inverse systems to compute Gorenstein covers. We propose new characterizations of the minimal Gorenstein cover and present a new algorithm for the effective computation of the variety of all minimal Gorenstein covers of A for low Gorenstein colength. Experimentation illustrates the practical behavior of the method.

1. Introduction

Given a local Artin $\mathbf{k}$-algebra $A = R/I$, with $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$, an interesting problem is to find how far is it from being Gorenstein. In [1], Ananthnarayan introduces for the first time the notion of Gorenstein colength, denoted by $\operatorname{gcl}(A)$, as the minimum of $\ell(G) - \ell(A)$ among all Gorenstein Artin $\mathbf{k}$-algebras $G = R/J$ mapping onto A. Two natural questions arise immediately:

Question A: How we can explicitly compute the Gorenstein colength of a given local Artin $\mathbf{k}$-algebra A?

Question B: Which are its minimal Gorenstein covers, that is, all Gorenstein rings G reaching the minimum $\operatorname{gcl}(A) = \ell(G) - \ell(A)$?

Ananthnarayan generalizes some results by Teter [14] and Huneke-Vraciu [10] and provides a characterization of rings of $\operatorname{gcl}(A) \leq 2$ in terms of the existence of certain self-dual ideals $\mathfrak{q} \in A$ with respect to the canonical module ω_A of A satisfying $\ell(A/\mathfrak{q}) \leq 2$. For more information on this, see [1] or [6, Section 4], for a reinterpretation in terms of inverse systems. Later on, Elias and Silva ([8]) address the problem of the colength from the perspective of Macaulay's inverse systems. In this setting, the goal is to find polynomials $F \in S$ such that $I^\perp \subset \langle F \rangle$ and $\ell(\langle F \rangle) - \ell(I^\perp)$ is minimal. Then the Gorenstein $\mathbf{k}$-algebra $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A. A precise characterization of such polynomials $F \in S$ is provided for $\operatorname{gcl}(A) = 1$ in [8] and for $\operatorname{gcl}(A) = 2$ in [6].

However, the explicit computation of the Gorenstein colength of a given ring A is not an easy task even for low colength - meaning $\operatorname{gcl}(A)$ equal or less than 2 - in the general case. For examples of computation of colength of certain families of rings, see [2] and [6].

* Partially supported by MTM2016-78881-P.

** Partially supported by MTM2016-78881-P, BES-2014-069364 and EEBB-I-17-12700.

*** Partially supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skodowska-Curie grant agreement No 813211 of POEMA.

2010 MSC: Primary 13H10; Secondary 13H15; 13P99.

On the other hand, if $\operatorname{gcl}(A) = 1$, the Teter variety introduced in [8, Proposition 4.2] is precisely the variety of all minimal Gorenstein covers of A and [8, Proposition 4.5] already suggests that a method to compute such covers is possible.

In this paper we address questions A and B by extending the previous definition of Teter variety of a ring of Gorenstein colength 1 to the variety of minimal Gorenstein covers $MGC(A)$ where A has arbitrary Gorenstein colength t. We use a constructive approach based on the integration method to compute inverse systems proposed by Mourrain in [11].

In section 2 we recall the basic definitions of inverse systems and introduce the notion of integral of an R-module M of S with respect to an ideal K of R, denoted by $\int_K M$. Section 3 links generators $F \in S$ of inverse systems $J^\perp$ of Gorenstein covers $G = R/J$ of $A = R/I$ with elements in the integral $\int_{\mathfrak{m}^t} I^\perp$, where $\mathfrak{m}$ is the maximal ideal of R and $t = \operatorname{gcl}(A)$. This relation is described in Proposition 3.6 and Theorem 3.8 sets the theoretical background to compute a $\mathbf{k}$-basis of the integral of a module extending Mourrain's integration method.

In section 4, Theorem 4.2 proves the existence of a quasi-projective sub-variety $MGC^n(A)$ whose set of closed points are associated to polynomials $F \in S$ such that $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A. Section 5 is devoted to algorithms: explicit methods to compute a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^t} I^\perp$ and $MGC(A)$ for colengths 1 and 2. Finally, in section 6 we provide several examples of the minimal Gorenstein covers variety and list the comptutation times of $MGC(A)$ for all analytic types of $\mathbf{k}$-algebras with $\operatorname{gcl}(A) \leq 2$ appearing in Poonen's classification in [12].

All algorithms appearing in this paper have been implemented in *Singular*, [4], and the library [5] for inverse system has also been used.

Acknowledgements: The second author wants to thank the third author for the opportunity to stay at INRIA Sophia Antipolis - Méditerranée (France) and his hospitality during her visit on the fall of 2017, where part of this project was carried out. This stay was financed by the Spanish Ministry of Economy and Competitiveness through the Estancias Breves programme (EEBB-I-17-12700).

2. Integrals and inverse systems

Let us consider the regular local ring $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$ over an arbitrary field $\mathbf{k}$, with maximal ideal $\mathfrak{m}$. Let $S = \mathbf{k}[y_1, \ldots, y_n]$ be the polynomial ring over the same field $\mathbf{k}$. Given $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$, we denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Recall that S can be given an R-module structure by contraction:

$$\begin{array}{ccc} R \times S & \longrightarrow & S \\ (x^\alpha, y^\beta) & \mapsto & x^\alpha \circ y^\beta = \left\{ \begin{array}{ll} y^{\beta-\alpha}, & \beta \geq \alpha; \\ 0, & \text{otherwise.} \end{array} \right. \end{array}$$

The Macaulay inverse system of $A = R/I$ is the sub-R-module $I^\perp = \{G \in S \mid I \circ G = 0\}$ of S. This provides the order-reversing bijection between $\mathfrak{m}$-primary ideals I of R and finitely generated sub-R-modules M of S described in Macaulay's duality. As for the reverse correspondence, given a sub-R-module M of S, the module $M^\perp$ is the ideal

$\operatorname{Ann}_{\mathrm{R}} M = \{f \in R \mid f \circ G = 0 \text{ for any } G \in M\}$ of R. Moreover, it characterizes zero-dimensional Gorenstein rings $G = R/J$ as those with cyclic inverse system $J^{\perp} = \langle F \rangle$, where $\langle F \rangle$ is the $\mathbf{k}$-vector space $\langle x^{\alpha} \circ F : |\alpha| \leq \deg F \rangle_{\mathbf{k}}$. For more details on this construction, see [8] and [6].

Consider an Artin local ring $A = R/I$ of socle degree s and inverse system $I^{\perp}$. We are interested in finding Artin local rings $R/\operatorname{Ann}_{\mathrm{R}} F$ that cover R/I, that is $I^{\perp} \subset \langle F \rangle$, but we also want to control how apart are those two inverse systems. In other words, given an ideal K, we want to find a Gorenstein cover $\langle F \rangle$ such that $K \circ \langle F \rangle \subset I^{\perp}$. Therefore it makes sense to think of an inverse operation to contraction.

Definition 2.1 (Integral of a module with respect to an ideal)**.** *Consider an R-submodule M of S. We define the integral of M with respect to the ideal K, denoted by $\int_K M$, as*

$$\int_K M = \{G \in S \mid K \circ G \subset M\}.$$

Note that the set $N = \{G \in S \mid K \circ G \subset M\}$ is, in fact, an R-submodule N of S endowed with the contraction structure. Indeed, given $G_1, G_2 \in N$ then $K \circ (G_1 + G_2) = K \circ G_1 + K \circ G_2 \subset M$, hence $G_1 + G_2 \in N$. For all $a \in R$ and $G \in N$ we have $K \circ (a \circ G) = aK \circ G = a \circ (K \circ G) \subset M$, hence $a \circ G \in N$.

Proposition 2.2. *Let K be an $\mathfrak{m}$-primary ideal of R and let M be a finitely generated sub-R-module of S. Then*

$$\int_K M = \left(KM^{\perp}\right)^{\perp}.$$

Proof. Let $G \in \left(KM^{\perp}\right)^{\perp}$. Then $\left(KM^{\perp}\right) \circ G = 0$, so $M^{\perp} \circ (K \circ G) = 0$. Hence $K \circ G \subset M$, i.e. $G \in \int_K M$. We have proved that $\left(KM^{\perp}\right)^{\perp} \subseteq \int_K M$. Now let $G \in \int_K M$. By definition, $K \circ G \subset M$, so $M^{\perp} \circ (K \circ G) = 0$ and hence $\left(M^{\perp}K\right) \circ G = 0$. Therefore, $G \in \left(M^{\perp}K\right)^{\perp}$. ◻

One of the key results of this paper is the effective computation of $\int_K M$ (see Algorithm 1). Last result gives a method for the computation of this module by computing two Macaulay duals. However, since computing Macaulay duals is expensive, Algorithm 1 avoids the computation of such duals.

Remark 2.3. The following properties hold: (i) Given $K \subset L$ ideals of R and M R-module, if $K \subset L$, then $\int_L M \subset \int_K M$. (ii) Given K ideal of R and $M \subset N$ R-modules, if $M \subset N$, then $\int_K M \subset \int_K N$. (iii) Given any R-module M, $\int_R M = M$.

The inclusion $K \circ \int_K M \subset M$ follows directly from the definition of integral. However, the equality does not hold:

Example 2.4. Let us consider $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$, $K = (x_1, x_2, x_3)$, $S = \mathbf{k}[y_1, y_2, y_3]$ and $M = \langle y_1y_2, y_3^3 \rangle$. We can compute Macaulay duals with the *Singular* library **Inverse-syst.lib**, see [5]. We get $\int_K M = \langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^4 \rangle$ by Proposition 2.2 and hence $K \circ \int_K M = \langle y_1, y_2, y_3^3 \rangle \subsetneq M$.

We also have the inclusion $M \subset \int_K K \circ M$. Indeed, for any $F \in M$, $K \circ F \subset K \circ M$ and hence $F \in \int_K K \circ M = \{G \in S \mid K \circ G \subset K \circ M\}$. Again, the equality does not hold.

Example 2.5. Using the same example as in Example 2.4, we get $K \circ M = \mathfrak{m} \circ \langle y_1y_2, y_3^3\rangle = \langle y_1, y_2, y_3^2\rangle$, and $\int_K (K \circ M) = \left(K(K \circ M)^{\perp}\right)^{\perp} = \langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^2\rangle \nsubseteq M$.

Remark 2.6. Note that if we integrate with respect to a principal ideal $K = (f)$ of R, then $\int_K M = \{G \in S \mid f \circ G \in M\}$. Hence in this case we will denote it by $\int_f M$.

In particular, if we consider a principal monomial ideal $K = (x^\alpha)$, then the expected equality for integrals

$$x^\alpha \circ \int_{x^\alpha} M = M$$

holds. Indeed, for any $m \in M$, take $G = y^\alpha m$. Since $x^\alpha \circ y^\alpha = 1$, then $x^\alpha \circ y^\alpha m = m$ and the equality is reached.

Remark 2.7. In general, $\int_{x^\alpha} x^\alpha \circ M \neq x^\alpha \circ \int_{x^\alpha} M$, hence the inclusion $M \subset \int_K K \circ M$ is not an equality even for principal monomial ideals. See Remark 2.9.

Let us now consider an even more particular case: the integral of a cyclic module $M = \langle F\rangle$ with respect to the variable x_i. Since the equality $x_i \circ \int_{x_i} M = M$ holds, there exists $G \in S$ such that $x_i \circ G = F$. This polynomial G is not unique because it can have any constant term with respect to x_i, that is $G = y_iF + p(y_1, \dots, \hat{y}_i, \dots, y_n)$. However, if we restrict to the non-constant polynomial we can define the following:

Definition 2.8 (i-primitive)**.** *The i-primitive of a polynomial $f \in S$ is the polynomial $g \in S$, denoted by $\int_i f$, such that*

(i) $x_i \circ g = f$,
(ii) $g|_{y_i=0} = 0$.

In [7], Elkadi and Mourrain proposed a definition of i-primitive of a polynomial in a zero-characteristic setting using the derivation structure instead of contraction. Therefore, we can think of the integral of a module with respect to an ideal as a generalization of their i-primitive.

Since we are considering the R-module structure given by contraction, the i-primitive is precisely

$$\int_i f = y_i f.$$

Indeed, $x_i \circ (y_if) = f$ and $(y_if)\ |_{y_i=0}= 0$, hence (i) and (ii) hold. Uniqueness can be easily proved. Consider g_1, g_2 to be i-primitives of f. Then $x_i \circ (g_1 - g_2) = 0$ and hence $g_1 - g_2 = p(y_1, \dots, \hat{y}_i, \dots, y_n)$. Clearly $(g_1 - g_2)|_{y_i=0} = p(y_1, \dots, \hat{y}_i, \dots, y_n)$. On the other hand, $(g_1 - g_2)|_{y_i=0} = g_1|_{y_i=0} - g_2|_{y_i=0} = 0$. Hence $p = 0$ and $g_1 = g_2$.

Remark 2.9. Note that, by definition, $x_k \circ \int_k f = f$. Any f can be decomposed in $f = f_1 + f_2$, where the first term is a multiple of y_k and the second has no appearances of this variable. Then

$$\int_k x_k \circ f = \int_k x_k \circ f_1 + \int_k x_k \circ f_2 = f_1 + \int_k 0 = f_1.$$

Therefore, in general,

$$f_1 = \int_k x_k \circ f \neq x_k \circ \int_k f = f.$$

However, for all $l \neq k$,

$$\int_l x_k \circ f = \frac{y_l f_1}{y_k} = x_k \circ \int_l f.$$

Let us now recall Theorem 7.36 of Elkadi-Mourrain in [7], which describes the elements of the inverse system $I^\perp$ up to a certain degree d. We define $\mathcal{D}_d = I^\perp \cap S_{\leq d}$, for any $1 \leq d \leq s$, where $s = \operatorname{socdeg}(A)$. Since $\mathcal{D}_s = I^\perp$, this result leads to an algorithm proposed by the same author to obtain a $\mathbf{k}$-basis of an inverse system. We rewrite the theorem using the contraction setting instead of derivation.

Theorem 2.10 (Elkadi-Mourrain). *Given an ideal $I = (f_1, \ldots, f_m)$ and $d > 1$. Let $\{b_1, \ldots, b_{t_{d-1}}\}$ be a $\mathbf{k}$-basis of $\mathcal{D}_{d-1}$. The polynomials of $\mathcal{D}_d$ with no constant term, i.e. no terms of degree zero, are of the form*

$$\text{(1)} \quad \Lambda = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^n \int_n b_j, \quad \lambda_j^k \in \mathbf{k},$$

such that

$$\text{(2)} \qquad \sum_{j=1}^{t_{d-1}} \lambda_j^k (x_l \circ b_j) - \sum_{j=1}^{t_{d-1}} \lambda_j^l (x_k \circ b_j) = 0, 1 \leq k < l \leq n,$$

and

$$\text{(3)} \qquad (f_i \circ \Lambda)(0) = 0, \text{ for } 1 \leq i \leq m.$$

See [11] or [7] for a proof.

3. Using integrals to obtain Gorenstein covers of Artin rings

Let us start by recalling the definitions of Gorenstein cover and Gorenstein colength of a local equicharacteristic Artin ring $A = R/I$ from [6]:

Definition 3.1. *We say that $G = R/J$, with $J = \operatorname{Ann_R} F$, is a Gorenstein cover of A if and only if $I^\perp \subset \langle F \rangle$. The Gorenstein colength of A is*

$$\operatorname{gcl}(A) = \min\{\ell(G) - \ell(A) \mid G \text{ is a Gorenstein cover of } A\}.$$

A Gorenstein cover G of an Artin ring A is minimal if $\ell(G) = \ell(A) + \operatorname{gcl}(A)$.

For all $F \in S$ defining a Gorenstein cover of A we consider the colon ideal K_F of R defined by

$$K_F = (I^\perp :_R \langle F \rangle).$$

In general, we do not know which are the ideals K_F that provide a minimal Gorenstein cover of a given ring. However, for a given colength, we do know a lot about the form of the ideals K_F associated to a polynomial F that reaches this minimum. In the following proposition, we summarize the basic results regarding ideals K_F from [6]:

Proposition 3.2. *Let $A = R/I$ be a local Artin algebra and $G = R/J$, with $J = \mathrm{Ann_R}\, F$, a minimal Gorenstein cover of A. Then,*

(i) $I^\perp = K_F \circ F$,
(ii) $\mathrm{gcl}(A) = \ell(R/K_F)$.

Moreover,

$$K_F = \begin{cases} R, & \text{if} \quad \mathrm{gcl}(A) = 0; \\ \mathfrak{m}, & \text{if} \quad \mathrm{gcl}(A) = 1; \\ (L_1, \dots, L_{n-1}, L_n^2), & \text{if} \quad \mathrm{gcl}(A) = 2, \end{cases}$$

where $L_1, \dots, L_n$ are suitable independent linear forms in R.

Remark 3.3. Note that whereas in the case of colength 1 the ideal K_F does not depend on the particular choice of F, this is no longer true for higher colengths. For colength higher that 2, things get more complicated since the K_F can even have different analytic type. The simplest example is colength 3, where we have 2 possible non-isomorphic K_F's: $(L_1, \dots, L_{n-1}, L_n^3)$ and $(L_1, \dots, L_{n-2}, L_{n-1}^2, L_{n-1}L_n, L_n^2)$. Therefore, although it is certainly true that $F \in \int_{K_F} I^\perp$, it will not be useful as a condition to check if A has a certain Gorenstein colength.

The dependency of the integral on F can be removed by imposing only the condition $F \in \int_{\mathfrak{m}^t} I^\perp$, for a suitable integer t. Later on we will see how to use this condition to find a minimal cover, but we first need to dig deeper into the structure of the integral of a module with respect to a power of the maximal ideal. The following result permits an iterative approach:

Lemma 3.4. *Let M be a finitely generated sub-R-module of S and $d \geq 1$, then*

$$\int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right) = \int_{\mathfrak{m}^d} M.$$

Proof. Let us prove first the inclusion $\int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right) \subseteq \int_{\mathfrak{m}^d} M$. Take $\Lambda \in \int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right)$, then $\mathfrak{m} \circ \Lambda \subseteq \int_{\mathfrak{m}^{d-1}} M$ and hence $\mathfrak{m}^d \circ \Lambda = \mathfrak{m}^{d-1} \circ (\mathfrak{m} \circ \Lambda) \subseteq M$. Therefore, $\Lambda \in \int_{\mathfrak{m}^d} M$. To prove the reverse inclusion, consider $\Lambda \in \int_{\mathfrak{m}^d} M$, that is, $\mathfrak{m}^{d-1} \circ (\mathfrak{m} \circ \Lambda) = \mathfrak{m}^d \circ \Lambda \subseteq M$. In other words, $\mathfrak{m} \circ \Lambda \subseteq \int_{\mathfrak{m}^{d-1}} M$ and $\Lambda \in \int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right)$. $\square$

Since $\int_{\mathfrak{m}^t} M$ is a finitely dimensional $\mathbf{k}$-vector space that can be obtained by integrating t times M with respect to $\mathfrak{m}$, we can also consider a basis of $\int_{\mathfrak{m}^t} M$ which is built by extending the previous basis at each step.

Definition 3.5. *Let M be a finitely generated sub-R-module of S. Given an integer t, we denote by h_i the dimension of the $\mathbf{k}$-vector space $\int_{\mathfrak{m}^i} M/\int_{\mathfrak{m}^{i-1}} M$, $i=1,\cdots,t$. An adapted $\mathbf{k}$-basis of $\int_{\mathfrak{m}^t} M/M$ is a $\mathbf{k}$-basis $\overline{F}^i_j$, $i=1,\cdots,t$, $j=1,\cdots,h_i$, of $\int_{\mathfrak{m}^t} M/M$ such that $F^i_1,\cdots,F^i_{h_i}\in\int_{\mathfrak{m}^i} M$ and their cosets in $\int_{\mathfrak{m}^i} M/\int_{\mathfrak{m}^{i-1}} M$ form a $\mathbf{k}$-basis, $i=1\cdots,t$. Let $A=R/I$ be an Artin ring, we denote by $\mathcal{L}_{A,t}$ the R-module $\int_{\mathfrak{m}^t} I^\perp/I^\perp$.*

The following proposition is meant to overcome the obstacle of non-uniqueness of the ideals K_F:

Proposition 3.6. *Given a ring $A=R/I$ of Gorenstein colength t and a minimal Gorenstein cover $G=R/\operatorname{Ann_R} F$ of A,*

- (i) *$F\in\int_{\mathfrak{m}^t} I^\perp$;*
- (ii) *for any $H\in\int_{\mathfrak{m}^t} I^\perp$, the condition $I^\perp\subset\langle H\rangle$ does not depend on the representative of the class $\overline{H}$ in $\mathcal{L}_{A,t}$.*

In particular, any $F'\in\int_{\mathfrak{m}^t} I^\perp$ such that $\overline{F'}=\overline{F}$ in $\mathcal{L}_{A,t}$ defines the same minimal Gorenstein cover $G=R/\operatorname{Ann_R} F$.

Proof. (i) By [6, Proposition 3.8], we have $\operatorname{gcl}(A)=\ell(R/K_F)$, where $K_F\circ F=I^\perp$ for any polynomial F that generates a minimal Gorenstein cover $G=R/\operatorname{Ann_R} F$ of A. From the definition of integral we have $F\in\int_{K_F} I^\perp$. Since $\ell(R/K_F)=t$, then $\operatorname{socdeg}(R/K_F)\leq t-1$. Indeed, the extremal case corresponds to the most expanded Hilbert function $\{1,1,\dots,1\}$, that is, a stretched algebra (see [13],[9]). Then $\operatorname{HF}_{R/K_F}(i)=0$, for any $i\geq t$, regardless of the particular form of K_F, and hence $\mathfrak{m}^t\subset K_F$. Therefore,

$$F\in\int_{K_F} I^\perp\subset\int_{\mathfrak{m}^t} I^\perp.$$

(ii) Consider a polynomial $H\in\int_{\mathfrak{m}^t} I^\perp$ such that $I^\perp\subset\langle H\rangle$. By [6, Proposition 3.8], $K_H\circ H=I^\perp$. Consider $H'\in\int_{\mathfrak{m}^t} I^\perp$ such that $\overline{H}=\overline{H'}$ in $\mathcal{L}_{A,t}$, so $H=H'+G$ for some $G\in I^\perp$. We want to prove that

$$K_H\circ H'+\mathfrak{m}\circ I^\perp=K_H\circ H+\mathfrak{m}\circ I^\perp=I^\perp. \tag{4}$$

The second equality is direct from $K_H\circ H=I^\perp$. Let us check the first. Take $h\circ H'+\mathfrak{m}\circ I^\perp\in K_H\circ H'+\mathfrak{m}\circ I^\perp$, with $h\in K_H\subset\mathfrak{m}$,

$$h\circ H'+\mathfrak{m}\circ I^\perp=h\circ H-h\circ G+\mathfrak{m}\circ I^\perp=h\circ H+\mathfrak{m}\circ I^\perp\subset K_H\circ H+\mathfrak{m}\circ I^\perp.$$

The same argument holds for the reverse inclusion. Therefore, Equation (4) holds and we can apply Nakayama's lemma to get $K_H\circ H'=I^\perp$. Hence $I^\perp\subset\langle H'\rangle$. In particular, $\langle H'\rangle=\langle H\rangle$. Indeed, since $H'=H-G$ and $\langle G\rangle\subset\langle I^\perp\rangle\subset\langle H\rangle$, then $H'\in\langle H\rangle+\langle G\rangle=\langle H\rangle$ and a similar argument gives $H\in\langle H'\rangle$. □

Observe that the proposition says that, although not all $F\in\int_{\mathfrak{m}^t} I^\perp$ correspond to covers $G=R/\operatorname{Ann_R} F$ of $A=R/I$, if F is actually a cover, then any $F'\in\int_{\mathfrak{m}^t} I^\perp$ such that $\overline{F'}=\overline{F}\in\mathcal{L}_{A,t}$ provides the exact same cover. That is, $\langle F'\rangle=\langle F\rangle$.

Corollary 3.7. *Let $A = R/I$ be an Artin ring of Gorenstein colength t and let $\{\overline{F}^i_j\}_{1\leq i\leq t, 1\leq j\leq h_i}$ be an adapted* $\mathbf{k}$*-basis of $\mathcal{L}_{A,t}$. Given a minimal Gorenstein cover $G = R/J$ there is a generator F of $J^\perp$ such that F can be written as*

$$F = a^1_1 F^1_1 + \cdots + a^1_{h_1} F^1_{h_1} + \cdots + a^t_1 F^t_1 + \cdots + a^t_{h_t} F^t_{h_t} \in \int_{\mathfrak{m}^t} I^\perp,\ a^j_i \in \mathbf{k}.$$

Proof. In $\mathcal{L}_{A,t}$ we have $\overline{F} = \sum_{i=1}^t \sum_{j=1}^{h_j} a^i_j \overline{F^i_j}$ and hence $F = \sum_{i=1}^t \sum_{j=1}^{h_i} a^i_j F^i_j + G$ with $G \in I^\perp$. By Proposition 3.6, any representative of the class $\overline{F}$ provides the same Gorenstein cover. In particular, we can take $G = 0$ and we are done. □

Our goal now is to compute the integrals of the inverse system with respect to powers of the maximal ideal. Rephrasing it in a more general manner: we want an effective computation of $\int_{\mathfrak{m}^k} M$, where $M \subset S$ is a sub-R-module of S and $k \geq 1$.

Recall that, via Macaulay's duality, we have $I^\perp = M$, where $I = \operatorname{Ann_R} M$ is an ideal in R. Therefore, the most natural approach is to integrate M in a similar way as I is integrated in Theorem 2.10 by Elkadi-Mourrain but removing the condition of orthogonality with respect to the generators of the ideal I (Equation (3) of Theorem 2.10). Without this restriction we will be allowed to go beyond the inverse system $I^\perp = M$ and up to the integral of M with respect to $\mathfrak{m}$. The proof we present is very similar to the proof of Theorem 7.36 in [11] but we reproduce it below for the sake of completeness and to show the use of the contraction structure.

Theorem 3.8. *Consider a sub-R-module M of S and let $\{b_1, \ldots, b_s\}$ be a* $\mathbf{k}$*-basis of M. Let $\Lambda \in S$ be a polynomial with no constant terms. Then $\Lambda \in \int_{\mathfrak{m}} M$ if and only if*

$$\Lambda = \sum_{j=1}^s \lambda^1_j \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^s \lambda^2_j \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^s \lambda^n_j \int_n b_j, \quad \lambda^k_j \in \mathbf{k}, \tag{5}$$

such that

$$\sum_{j=1}^s \lambda^k_j (x_l \circ b_j) - \sum_{j=1}^s \lambda^l_j (x_k \circ b_j) = 0, 1 \leq k < l \leq n. \tag{6}$$

Proof. Consider a polynomial Λ in $\int_{\mathfrak{m}} M$ with no constant term. Observe that we have a unique decomposition $\Lambda = \sum_{l=1}^n \Lambda_l$ such that Λ_l is a polynomial in $\mathbf{k}[y_l, \ldots, y_n] \backslash \mathbf{k}[y_{l+1}, \ldots, y_n]$. By definition, $x_1 \circ \Lambda_1 = x_1 \circ \Lambda$ is in M, hence $x_1 \circ \Lambda_1 = \sum_{j=1}^s \lambda^1_j b_j$ for some unique scalars λ^1_j in $\mathbf{k}$. Note that each Λ_l is a multiple of y_l. By Remark 2.9,

$$\Lambda_1 = \int_1 x_1 \circ \Lambda_1 = \sum_{j=1}^s \lambda^1_j \int_1 b_j.$$

Again, $x_2 \circ \Lambda = x_2 \circ \Lambda_1 + x_2 \circ \Lambda_2$ is in M, hence there exist unique scalars λ^2_j in $\mathbf{k}$ such that $x_2 \circ \Lambda = \sum_{j=1}^s \lambda^2_j b_j$. It can be checked that $\int_2 x_2 \circ \Lambda_1 = \Lambda_1 - \Lambda_1 |_{y_2=0}$. Then

$$\Lambda_2 = \int_2 x_2 \circ \Lambda_2 = \int_2 x_2 \circ \Lambda - \int_2 x_2 \circ \Lambda_1 = \sum_{j=1}^s \lambda^2_j \int_2 b_j - \left(\Lambda_1 - \Lambda_1|_{y_2=0}\right).$$

Similarly, for any $1 \leq l \leq n$, we can obtain

$$\Lambda_l = \sum_{j=1}^{s} \lambda_j^l \int_l b_j - \left(\sigma_{l-1} - \sigma_{l-1} \mid_{y_l=0}\right), \tag{7}$$

where

$$\sigma_k = \sum_{l=1}^{k} \Lambda_l = \sum_{j=1}^{s} \lambda_j^1 \int_1 b_j \mid_{y_2=\cdots=y_k=0} + \sum_{j=1}^{s} \lambda_j^2 \int_2 b_j \mid_{y_3=\cdots=y_k=0} + \cdots + \sum_{j=1}^{s} \lambda_j^k \int_k b_j, \tag{8}$$

for any $1 \leq k \leq n$ and $\sigma_0 = 0$.

Since $\Lambda = \sigma_n$, we get (5). We want to prove now that (6) holds. Since $\Lambda_l \in \mathbf{k}[y_l, \ldots, y_n]$, then $x_k \circ \Lambda_l = 0$ for $1 \leq k < l \leq n$. Hence contracting (7) first by x_k and then by x_l we get

$$\sum_{j=1}^{s} \lambda_j^l (x_k \circ b_j) = x_l \circ (x_k \circ \sigma_{l-1}). \tag{9}$$

On one hand, for $k < l$, $x_k \circ \sigma_{l-1} = x_k \circ (\sum_{i=1}^{k} \Lambda_i) = x_k \circ \sigma_k$. On the other hand, when contracting (8) by x_k, the first $k-1$ terms vanish:

$$x_k \circ \sigma_k = \sum_{j=1}^{s} \lambda_j^k \left(x_k \circ \int_k b_j \right) = \sum_{j=1}^{s} \lambda_j^k b_j.$$

Therefore, we can rewrite (9) as $\sum_{j=1}^{s} \lambda_j^l (x_k \circ b_j) = \sum_{j=1}^{s} \lambda_j^k (x_l \circ b_j)$, hence (6) is satisfied.

Conversely, we want to know if every element of the form (5) satisfying (6) is in $\int_{\mathfrak{m}} M$. It is enough to prove that $x_k \circ \Lambda \in M$ for any $1 \leq k \leq n$. Let us then contract (5) by x_k for any $1 \leq k \leq n$:

$$x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k b_j \mid_{y_{k+1}=\cdots=y_n=0} + \sum_{j=1}^{s} \lambda_j^{k+1} \int_{k+1} x_k \circ b_j \mid_{y_{k+2}=\cdots=y_n=0} + \cdots + \sum_{j=1}^{s} \lambda_j^n \int_n x_k \circ b_j.$$

The l-primitive of (6), for any $k < l \leq n$, gives

$$\sum_{j=1}^{s} \lambda_j^k \int_l x_l \circ b_j = \sum_{j=1}^{s} \lambda_j^l \int_l x_k \circ b_j.$$

Hence

$$x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k \left(b_j \mid_{y_{k+1}=\cdots=y_n=0} + \int_{k+1} x_{k+1} \circ b_j \mid_{y_{k+2}=\cdots=y_n=0} + \cdots + \int_n x_n \circ b_j \right).$$

It can be proved that the expression in the parenthesis is exactly b_j for any $1 \leq j \leq n$, hence $x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k b_j$ and we are done. $\square$

From the previous theorem and Lemma 3.4 the next corollary follows directly.

Corollary 3.9. *Consider a sub-R-module M of S and $d \geq 1$. Let $\{b_1, \dots, b_{t_{d-1}}\}$ be a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^{d-1}} M$ and let Λ be a polynomial with no constant terms. Then $\Lambda \in \int_{\mathfrak{m}^d} M$ if and only if it is of the form*

$$\Lambda = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^n \int_n b_j, \quad \lambda_j^k \in \mathbf{k}, \tag{10}$$

such that

$$\sum_{j=1}^{t_{d-1}} \lambda_j^k (x_l \circ b_j) - \sum_{j=1}^{t_{d-1}} \lambda_j^l (x_k \circ b_j) = 0, \quad 1 \leq k < l \leq n. \tag{11}$$

Remark 3.10. Note that, using the notations of Theorem 2.10, it can be proved that

$$\mathcal{D}_d = I^\perp \cap \int_{\mathfrak{m}} \mathcal{D}_{d-1},$$

for any $1 < d \leq s$. Indeed, Theorem 2.10 says that any element $\Lambda \in \mathcal{D}_d$ is of the form of Equation (10), and because of Corollary 3.9, we know that it satisfies Equation (11). Hence, by Theorem 3.8, $\Lambda \in \int_{\mathfrak{m}} \mathcal{D}_{d-1}$. Since $\Lambda \in \mathcal{D}_d = I^\perp \cap S_{\leq d}$, then $\Lambda \in I^\perp \cap \int_{\mathfrak{m}} \mathcal{D}_{d-1}$. Conversely, any element Λ in $\left(\int_{\mathfrak{m}} \mathcal{D}_{d-1}\right) \cap I^\perp$ satisfies, in particular, $\mathfrak{m} \circ \Lambda \subseteq \mathcal{D}_{d-1} = I^\perp \cap S_{\leq d-1}$. Therefore $\deg(\mathfrak{m} \circ \Lambda) \leq d-1$ and hence $\deg \Lambda \leq d$. Since $\Lambda \in I^\perp$, then $\Lambda \in I^\perp \cap S_{\leq d} = \mathcal{D}_d$.

We end this section by considering the low Gorenstein colength cases.

3.1. **Teter rings.** Let us remind that Teter rings are those $A = R/I$ such that $A \cong G/\operatorname{soc}(G)$ for some Artin Gorenstein ring G. In [8], the authors prove that $\operatorname{gcl}(A) = 1$ whenever $\operatorname{embdim}(A) \geq 2$. They are a special case to deal with because the K_F associated to any generator $F \in S$ of a minimal cover is always the maximal ideal. We provide some additional criteria to characterize such rings:

Proposition 3.11. *Let $A = R/I$ be a non-Gorenstein local Artin ring of socle degree $s \geq 1$ and let $\{\overline{F}_j\}_{1 \leq j \leq h}$ be an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$. Then $\operatorname{gcl}(A) = 1$ if and only if there exist a polynomial $F = \sum_{j=1}^h a_j F_j \in \int_{\mathfrak{m}} I^\perp$, $a_j \in \mathbf{k}$, such that $\dim_{\mathbf{k}}(\mathfrak{m} \circ F) = \dim_{\mathbf{k}} I^\perp$.*

Proof. The first implication is straightforward from Corollary 3.7 and Teter rings characterization in [8]. Reciprocally, if $F \in \int_{\mathfrak{m}} I^\perp$, then $\mathfrak{m} \circ F \subset I^\perp$ by definition, and from the equality of dimensions, it follows that $\mathfrak{m} \circ F = I^\perp$. Therefore, $0 < \operatorname{gcl}(A) \leq \ell(R/\mathfrak{m}) = 1$ and we are done. $\square$

Example 3.12. Recall Example 2.4 with $I^\perp = \langle y_1 y_2, y_3^3 \rangle$ and $\int_{\mathfrak{m}} I^\perp = \langle y_1^2, y_1 y_2, y_1 y_3, y_2^2, y_2 y_3, y_3^4 \rangle$. Then $\overline{y_1^2}, \overline{y_1 y_3}, \overline{y_2^2}, \overline{y_2 y_3}, \overline{y_3^4}$ is a $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$. As a consequence of Proposition 3.11, A is Teter if and only if there exists a polynomial

$$F = a_1 y_1^2 + a_2 y_1 y_3 + a_3 y_2^2 + a_4 y_2 y_3 + a_5 y_3^4$$

such that $\mathfrak{m} \circ F = I^\perp$. But $\mathfrak{m} \circ F = \langle a_1 y_1 + a_2 y_3, a_3 y_2 + a_4 y_3, a_2 y_1 + a_4 y_2 + a_5 y_3^3 \rangle$ and clearly $y_1 y_2$ does not belong here. Therefore, $\operatorname{gcl}(A) > 1$.

3.2. **Gorenstein colength 2.** By [6], we know that an Artin ring A of socle degree s is of Gorenstein colength 2 if and only if there exists a polynomial F of degree $s+1$ or $s+2$ such that $K_F \circ F = I^\perp$, where $K_F = (L_1, \dots, L_{n-1}, L_n^2)$ and $L_1, \dots, L_n$ are suitable independent linear forms.

Observe that a completely analogous characterization to the one we did for Teter rings is not possible. If $A = R/I$ has Gorenstein colength 2, by Corollary 3.7, there exists $F = \sum_{i=1}^2 \sum_{j=1}^{h_i} a_j^i F_j^i \in \int_{\mathfrak{m}^2} I^\perp$, where $\{\overline{F_j^i}\}_{1\le i\le 2, 1\le j\le h_i}$ is a $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$, that generates a minimal Gorenstein cover of A and then trivially $I^\perp \subset \langle F \rangle$. However, the reverse implication is not true.

Example 3.13. Consider $A = R/\mathfrak{m}^3$, where R is the ring of power series in 2 variables, and consider $F = y_1^2 y_2^2$. It is easy to see that $F \in \int_{\mathfrak{m}^2} I^\perp = S_{\le 4}$ and $I^\perp \subset \langle F \rangle$. However, it can be proved that $\operatorname{gcl}(A) = 3$ using [2, Corollary 3.3]. Note that $K_F = \mathfrak{m}^2$ and hence $\ell(R/K_F) = 3$.

Therefore, given $F \in \int_{\mathfrak{m}^2} I^\perp$, the condition $I \subset \langle F \rangle$ is not sufficient to ensure that $\operatorname{gcl}(A) = 2$. We must require that $\ell(R/K_F) = 2$ as well.

Proposition 3.14. *Given a non-Gorenstein non-Teter local Artin ring $A = R/I$, $\operatorname{gcl}(A) = 2$ if and only if there exist a polynomial $F = \sum_{i=1}^2 \sum_{j=1}^{h_i} a_j^i F_j^i \in \int_{\mathfrak{m}^2} I^\perp$ such that $\{\overline{F_j^i}\}_{1\le i\le 2, 1\le j\le h_i}$ is an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$ and $(L_1, \dots, L_{n-1}, L_n^2) \circ F = I^\perp$ for suitable independent linear forms $L_1, \dots, L_n$.*

Proof. We will only prove that if F satisfies the required conditions, then $\operatorname{gcl}(A) = 2$. By definition of K_F, if $(L_1, \dots, L_{n-1}, L_n^2) \circ F = I^\perp$, then $(L_1, \dots, L_{n-1}, L_n^2) \subseteq K_F$. Again by [6], $\operatorname{gcl}(A) \le \ell(R/K_F)$ and hence $\operatorname{gcl}(A) \le \ell\left(R/(L_1, \dots, L_{n-1}, L_n^2)\right) = 2$. Since $\operatorname{gcl}(A) \ge 2$ by hypothesis, then $\operatorname{gcl}(A) = 2$. The converse implication follows from Proposition 3.2. $\square$

Example 3.15. Recall the ring $A = R/I$ in Example 3.12. Since

$$\int_{\mathfrak{m}^2} I^\perp = \langle y_1^3, y_1^2 y_2, y_1 y_2^2, y_2^3, y_1^2 y_3, y_1 y_2 y_3, y_2^2 y_3, y_1 y_3^2, y_2 y_3^3, y_3^5 \rangle$$

and $\operatorname{gcl}(A) > 1$, its Gorenstein colength is 2 if and only if there exist some

$$F \in \langle y_1^2, y_1 y_2, y_1 y_3, y_2^2, y_2 y_3, y_3^4, y_1^3, y_1^2 y_2, y_1 y_2^2, y_2^3, y_1^2 y_3, y_1 y_2 y_3, y_2^2 y_3, y_1 y_3^2, y_2 y_3^3, y_3^5 \rangle_{\mathbf{k}}$$

such that $(L_1, \dots, L_{n-1}, L_n^2) \circ F = I^\perp$. Consider $F = y_3^4 + y_1^2 y_2$, then

$$(x_1, x_2^2, x_3) \circ F = \langle x_1 \circ F, x_2^2 \circ F, x_3 \circ F \rangle = \langle y_1 y_2, y_3^3 \rangle$$

and hence $\operatorname{gcl}(A) = 2$.

4. Minimal Gorenstein covers varieties

We are now interested in providing a geometric interpretation of the set of all minimal Gorenstein covers $G = R/J$ of a given local Artin $\mathbf{k}$-algebra $A = R/I$. From now on, we will assume that $\mathbf{k}$ is an algebraically closed field. The following result is well known and it is an easy linear algebra exercise.

Lemma 4.1. *Let $\varphi_i : \mathbf{k}^a \longrightarrow \mathbf{k}^b$, $i = 1\cdots,r$, be a family of Zariski continuous maps. Then the function $\varphi^* : \mathbf{k}^a \longrightarrow \mathbb{N}$ defined by $\varphi^*(z) = \dim_{\mathbf{k}}\langle\varphi_1(z),\cdots,\varphi_r(z)\rangle_{\mathbf{k}}$ is lower semicontinous, i.e. for all $z_0 \in \mathbf{k}^a$ there is a Zariski open set $z_0 \in U \subset \mathbf{k}^a$ such that for all $z \in U$ it holds $\varphi^*(z) \geq \varphi^*(z_0)$.*

Theorem 4.2. *Let $A = R/I$ be an Artin ring of Gorenstein colength t. There exists a quasi-projective sub-variety $MGC^n(A)$, $n = \dim(R)$, of $\mathbb{P}_{\mathbf{k}}(\mathcal{L}_{A,t})$ whose set of closed points are the points $[\overline{F}]$, $\overline{F} \in \mathcal{L}_{A,t}$, such that $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A.*

Proof. Let E be a sub-$\mathbf{k}$-vector space of $\int_{\mathfrak{m}^t} I^\perp$ such that

$$\int_{\mathfrak{m}^t} I^\perp \cong E \oplus I^\perp,$$

we identify $\mathcal{L}_{A,t}$ with E. From Proposition 3.6, for all minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ we may assume that $F \in E$. Given $F \in E$, the quotient $G = R/\operatorname{Ann_R} F$ is a minimal cover of A if and only if the following two numerical conditions hold:

(1) $\dim_{\mathbf{k}}(\langle F\rangle) = \dim_{\mathbf{k}} A + t$, and
(2) $\dim_{\mathbf{k}}(I^\perp + \langle F\rangle) = \dim_{\mathbf{k}}\langle F\rangle$.

Define the family of Zariski continuous maps $\{\varphi_\alpha\}_{|\alpha|\leq \deg F}$, $\alpha \in \mathbb{N}^n$, where

$$\begin{array}{cccc} \varphi_\alpha : & E & \longrightarrow & E \\ & F & \longmapsto & x^\alpha \circ F \end{array}$$

In particular, $\varphi_0 = Id_R$. We write

$$\begin{array}{cccc} \varphi^* : & E & \longrightarrow & \mathbb{N} \\ & F & \longmapsto & \dim_{\mathbf{k}}\langle x^\alpha \circ F, |\alpha| \leq \deg F\rangle_{\mathbf{k}} \end{array}$$

Note that $\varphi^*(F) = \dim_{\mathbf{k}}\langle F\rangle$ and, by Lemma 4.1, φ^* is a lower semicontinuous map. Hence $U_1 = \{F \in E \mid \dim_{\mathbf{k}}\langle F\rangle \geq \dim_{\mathbf{k}} A + t\}$ is an open Zariski set in E. Using the same argument, $U_2 = \{F \in E \mid \dim_{\mathbf{k}}\langle F\rangle \geq \dim_{\mathbf{k}} A + t + 1\}$ is also an open Zariski set in E and hence $Z_1 = E\backslash U_2$ is a Zariski closed set such that $\dim_{\mathbf{k}}\langle F\rangle \leq \dim_{\mathbf{k}} A + t$ for any $F \in Z_1$. Then $Z_1 \cap U_1 = \{F \in E \mid \dim_{\mathbf{k}}\langle F\rangle = \dim_{\mathbf{k}} A + t\}$ is a locally closed set.

Let $G_1,\cdots,G_e$ be a $\mathbf{k}$-basis of $I^\perp$ and consider the constant map

$$\begin{array}{cccc} \psi_i : & E & \longrightarrow & E \\ & F & \longmapsto & G_i \end{array}$$

for any $i = 1,\cdots,e$. By Lemma 4.1,

$$\begin{array}{cccc} \psi^* : & E & \longrightarrow & \mathbb{N} \\ & F & \longmapsto & \dim_{\mathbf{k}}\langle\{x^\alpha \circ F\}_{|\alpha|\leq \deg F}, G_1,\ldots,G_e\rangle_{\mathbf{k}} = \dim_{\mathbf{k}}\left(\langle F\rangle + I^\perp\right) \end{array}$$

is a lower semicontinuous map. Using an analogous argument, we can prove that $T = \{F \in E \mid \dim_{\mathbf{k}}(I^\perp + \langle F\rangle) = \dim_{\mathbf{k}} A + t\}$ is a locally closed set. Therefore,

$$W = (Z_1 \cap U_1) \cap T = \{F \in E \mid \dim_{\mathbf{k}} A + t = \dim_{\mathbf{k}}(I^\perp + \langle F\rangle) = \dim_{\mathbf{k}}\langle F\rangle\}$$

is a locally closed subset of E whose set of closed points are all the F in E satisfying (1) and (2), i.e. defining a minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ of A.

Moreover, since $\langle F\rangle = \langle \lambda F\rangle$ for any $\lambda \in \mathbf{k}^*$, conditions (1) and (2) are invariant under the multiplicative action of $\mathbf{k}^*$ on F and hence $MGC^n(A) = \mathbb{P}_{\mathbf{k}}(W) \subset \mathbb{P}_{\mathbf{k}}(E) = \mathbb{P}_{\mathbf{k}}(\mathcal{L}_{A,t})$. $\square$

Recall that the embedding dimension of A is $\operatorname{embdim}(A) = \dim_{\mathbf{k}} \mathfrak{m}/(\mathfrak{m}^2 + I)$.

Proposition 4.3. *Let G be a minimal Gorenstein cover of A. Then*

$$\operatorname{embdim}(G) \leq \tau(A) + \operatorname{gcl}(A) - 1.$$

Proof. Set $A = R/I$ such that $\operatorname{embdim}(A) = \dim R = n$. Consider the power series ring R' of dimension $n+t$ over $\mathbf{k}$ for some $t \geq 0$ such that $G = R'/J'$ with $\operatorname{embdim}(G) = \dim R'$. See [6] for more details on this construction. We denote by $\mathfrak{m}$ and $\mathfrak{m}'$ the maximal ideals of R and R', respectively, and consider $K_{F'} = (I^\perp :_{R'} F')$. From Proposition 3.2.$(i)$, it is easy to deduce that $K_{F'}/(\mathfrak{m}K_{F'} + J') \simeq I^\perp/(\mathfrak{m} \circ I^\perp)$. Hence $\tau(A) = \dim_{\mathbf{k}} K_{F'}/(\mathfrak{m}K_{F'} + J')$ by [8, Proposition 2.6]. Then

$$\operatorname{embdim}(G) + 1 = \dim_{\mathbf{k}} R'/(\mathfrak{m}')^2 \leq \dim_{\mathbf{k}} R'/(\mathfrak{m}K_{F'} + J') = \operatorname{gcl}(A) + \tau(A),$$

where the last equality follows from Proposition 3.2.(ii). $\square$

Definition 4.4. *Given an Artin ring $A = R/I$, the variety $MGC(A) = MGC^n(A)$, with $n = \tau(A) + \operatorname{gcl}(A) - 1$, is called the minimal Gorenstein cover variety associated to A.*

Remark 4.5. Let us recall that in [6] we proved that for low Gorenstein colength of A, i.e. $\operatorname{gcl}(A) \leq 2$, $\operatorname{embdim}(G) = \operatorname{embdim}(A)$ for any minimal Gorenstein cover G of A. In this situation we can consider $MGC(A)$ as the variety $MGC^n(A)$ with $n = \operatorname{embdim}(A)$.

Observe that this notion of minimal Gorenstein cover variety generalizes the definition of Teter variety introduced in [8], which applies only to rings of Gorenstein colength 1, to any arbitrary colength.

5. Computing $MGC(A)$ for low Gorenstein colength

In this section we provide algorithms and examples to compute the variety of minimal Gorenstein covers of a given ring A whenever its Gorenstein colength is 1 or 2. These algorithms can also be used to decide whether a ring has colength greater than 2, since it will correspond to empty varieties.

To start with, we provide the auxiliar algorithm to compute the integral of $I^\perp$ with respect to the t-th power of the maximal ideal of R. If there exist polynomials defining minimal Gorenstein covers of colength t, they must belong to this integral.

5.1. **Computing integrals of modules.** Consider a $\mathbf{k}$-basis $\mathbf{b} = (b_1, \dots, b_t)$ of a finitely generated sub-R-module M of S and consider $x_k \circ b_i = \sum_{j=1}^t a_j^i b_j$, for any $1 \leq i \leq t$ and $1 \leq k \leq n$. Let us define matrices $U_k = (a_j^i)_{1 \leq j, i \leq t}$ for any $1 \leq k \leq n$. Note that

$$(x_k \circ b_1 \cdots x_k \circ b_t) = (b_1 \cdots b_t) \begin{pmatrix} a_1^1 & \dots & a_1^t \\ \vdots & & \vdots \\ a_t^1 & \dots & a_t^t \end{pmatrix}.$$

Now consider any element $h \in M$. Then

$$x_k \circ h = x_k \circ \sum_{i=1}^{t} h_i b_i = \sum_{i=1}^{t} (x_k \circ h_i b_i) = \sum_{i=1}^{t} (x_k \circ b_i) h_i =$$
$$= (x_k \circ b_1 \cdots x_k \circ b_t) \begin{pmatrix} h_1 \\ \vdots \\ h_t \end{pmatrix} = (b_1 \cdots b_t)\, U_k \begin{pmatrix} h_1 \\ \vdots \\ h_t \end{pmatrix},$$

where $h_1, \dots, h_t \in \mathbf{k}$.

Definition 5.1. *Let U_k, $1 \le k \le n$, be the square matrix of order t such that*

$$x_k \circ h = \mathbf{b}\, U_k\, \mathbf{h}^t,$$

where $\mathbf{h} = (h_1, \dots, h_t)$ for any $h \in M$, with $h = \sum_{i=1}^{t} h_i b_i$. We call U_k the contraction matrix of M with respect to x_k associated to a $\mathbf{k}$-basis $\mathbf{b}$ of M.

Remark 5.2. Since $x_k x_l \circ h = x_l x_k \circ h$ for any $h \in M$, we have $U_k U_l = U_l U_k$, with $1 \le k < l \le n$.

In [11], Mourrain provides an effective algorithm based on Theorem 2.10 that computes, along with a $\mathbf{k}$-basis of the inverse system $I^\perp$ of an ideal I of R, the contraction matrices $U_1, \dots, U_n$ of $I^\perp$ associated to that basis.

Example 5.3. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = \mathfrak{m}^2$. Then $\{1, y_1, y_2\}$ is a $\mathbf{k}$-basis of $I^\perp$ and U_1, U_2 are its contraction matrices with respect to x_1, x_2, respectively:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we provide a modified algorithm based on Theorem 3.8 that computes the integral of a finitely generated sub-R-module M with respect to the maximal ideal. The algorithm can use the output of Mourrain's integration method as initial data: a $\mathbf{k}$-basis of $I^\perp$ and the contraction matrices associated to this basis.

Algorithm 1 Compute a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} M$ and its contraction matrices

Input: $D = b_1, \ldots, b_t$ $\mathbf{k}$-basis of M;
$U_1, \ldots, U_n$ contraction matrices of M associated to the $\mathbf{k}$-basis D.
Output: $D = b_1, \ldots, b_t, b_{t+1}, \ldots, b_{t+h}$ $\mathbf{k}$-basis of $\int_{\mathfrak{m}} M$;
$U_1', \ldots, U_n'$ contraction matrices of $\int_{\mathfrak{m}} M$ associated to the $\mathbf{k}$-basis D.
Steps:

(1) Set $\lambda_i = (\lambda_1^i \cdots \lambda_t^i)^t$, for any $1 \leq i \leq n$. Solve the system of equations

$$U_k \lambda_l - U_l \lambda_k = 0 \text{ for any } 1 \leq k < l \leq n. \tag{12}$$

(2) Consider a system of generators $\mathbf{H}_1, \ldots, \mathbf{H}_m$ of the solutions of Equation (12).
(3) For any $\mathbf{H}_i = [\lambda_1, \ldots, \lambda_n]$, $1 \leq i \leq m$, define the associated polynomial

$$\Lambda_{\mathbf{H}_i} = \sum_{k=1}^{n} \left(\sum_{j=1}^{t} \lambda_j^k \int_k b_j|_{y_{k+1}=\cdots=y_n=0} \right).$$

(4) If $\Lambda_{\mathbf{H}_1} \notin \langle D \rangle_{\mathbf{k}}$, then $b_{t+1} := \Lambda_{\mathbf{H}_1}$ and $D = D, b_{t+1}$. Repeat the procedure for $\Lambda_{\mathbf{H}_2}, \ldots, \Lambda_{\mathbf{H}_m}$.
(5) Set h as the number of new elements in D.
(6) Define square matrices U_k' of order $t+h$ and set $U_k'[i] = U_k[i]$ for $1 \leq i \leq t$.
(7) Compute $x_k \circ b_i = \sum_{j=1}^{t} \mu_j^i b_j$ for $t+1 \leq i \leq t+h$ and set

$$U_k'[i] = \begin{pmatrix} \mu_1^i & \cdots & \mu_t^i & 0 & \cdots & 0 \end{pmatrix}^t.$$

Remark 5.4. Observe that the classes in $\int_{\mathfrak{m}} M/M$ of the output $b_{t+1}, \ldots, b_{t+h}$ of Algorithm 1 form a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} M/M$. Moreover, since the algorithm returns the contraction matrices of $\int_{\mathfrak{m}} M$, we can iterate the procedure in order to obtain a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^k} M$ for any $k \geq 1$. By construction, the elements of this $\mathbf{k}$-basis that do not belong to M form an adapted $\mathbf{k}$-basis of $\int_{\mathfrak{m}^k} M/M$.

Example 5.5. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = \mathfrak{m}^2$. Then $\{1, y_1, y_2, y_2^2, y_1y_2, y_1^2\}$ is a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} I^\perp = S_{\leq 2}$ with the following contraction matrices:

$$U_1' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_2' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.2. **Computing $MGC(A)$ for Teter rings.** The following algorithm provides a method to decide whether a non-Gorenstein ring $A = R/I$ has colength 1 and, if this is the case, it explicitly computes its $MGC(A)$.

Let us consider a non-Gorenstein local Artin ring $A = R/I$ of socle degree s. Fix a $\mathbf{k}$-basis $b_1, \ldots, b_t$ of $I^\perp$ and consider a polynomial $F = \sum_{j=1}^{h} a_j F_j \in \int_{\mathfrak{m}} I^\perp$, where $\overline{F}_1, \ldots, \overline{F}_h$

is an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$. According to Proposition 3.11, F corresponds to a minimal Gorenstein cover if and only if $\dim_{\mathbf{k}}(\mathfrak{m}\circ F)=t$. Therefore, we want to know for which values of $a_1,\dots,a_h$ this equality holds.

Note that $\deg F\leq s+1$ and $x_kx_l\circ F=x_lx_k\circ F$. Then $\mathfrak{m}\circ F=\langle x^\alpha\circ F:1\leq|\alpha|\leq s+1\rangle_{\mathbf{k}}$. Moreover, by definition of F, each $x^\alpha\circ F\in I^\perp$, hence $x^\alpha\circ F=\sum_{j=1}^t\mu_\alpha^j b_j$ for some $\mu_\alpha^j\in\mathbf{k}$.

Consider the matrix $A=(\mu_\alpha^j)_{1\leq|\alpha|\leq s+1,\,1\leq j\leq t}$, whose rows are the contractions $x^\alpha\circ F$ expressed in terms of the $\mathbf{k}$-basis $b_1,\dots,b_t$ of $I^\perp$. The rows of A are a system of generators of $\mathfrak{m}\circ F$ as $\mathbf{k}$-vector space, hence $\dim_{\mathbf{k}}(\mathfrak{m}\circ F)<t$ if and only if all order t minors of A vanish. Let $\mathfrak{a}$ be the ideal generated by all order t minors $p_1,\dots,p_r$ of A. Note that the entries of matrix A are homogeneous polynomials of degree 1 in $\mathbf{k}[a_1,\dots,a_h]$. Hence $\mathfrak{a}$ is generated by homogeneous polynomials of degree t in $\mathbf{k}[a_1,\dots,a_h]$. Therefore, we can view the projective algebraic set

$$\mathbb{V}_+(\mathfrak{a})=\{[a_1:\dots:a_h]\in\mathbb{P}_{\mathbf{k}}^{h-1}\mid p_i(a_1,\dots,a_h)=0,\,1\leq i\leq r\},$$

as the set of all points that do not correspond to Teter covers. We just proved the following result:

Theorem 5.6. *Let $A=R/I$ be an Artin ring with* $\operatorname{gcl}(A)=1$, *$h=\dim_{\mathbf{k}}\mathcal{L}_{A,1}$ and $\mathfrak{a}$ be the ideal of minors previously defined. Then*

$$MGC(A)=\mathbb{P}_{\mathbf{k}}^{h-1}\backslash\mathbb{V}_+(\mathfrak{a}).$$

Moreover, for any non-Gorenstein Artin ring A, $\operatorname{gcl}(A)=1$ *if and only if* $\mathfrak{a}\neq 0$.

Proof. The first part is already proved. On the other hand, if $\mathfrak{a}=0$, then $\mathbb{V}_+(\mathfrak{a})=\mathbb{P}_{\mathbf{k}}^{h-1}$ and $MGC(A)=\emptyset$. In other words, there exist no Teter covers, hence $\operatorname{gcl}(A)>1$. □

Algorithm 2 Compute the Teter variety of $A=R/I$ with $n\geq 2$

Input: s socle degree of $A=R/I$;
$b_1,\dots,b_t$ $\mathbf{k}$-basis of the inverse system $I^\perp$;
$F_1,\dots,F_h$ adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$;
$U_1,\dots,U_n$ contraction matrices of $\int_{\mathfrak{m}}I^\perp$.
Output: ideal $\mathfrak{a}$ such that $MGC(A)=\mathbb{P}_{\mathbf{k}}^{h-1}\backslash\mathbb{V}_+(\mathfrak{a})$.
Steps:

(1) Set $F=a_1F_1+\cdots+a_hF_h$ and $\mathbf{F}=(a_1,\dots,a_h)^t$, where $a_1,\dots,a_h$ are variables in $\mathbf{k}$.
(2) Build matrix $A=\left(\mu_j^\alpha\right)_{1\leq|\alpha|\leq s+1,1\leq j\leq t}$, where

$$U^\alpha\mathbf{F}=\sum_{j=1}^t\mu_j^\alpha b_j,\quad U^\alpha=U_1^{\alpha_1}\cdots U_n^{\alpha_n}.$$

(3) Compute the ideal $\mathfrak{a}$ generated by all minors of order t of the matrix A.

With the following example we show how to apply and interpret the output of the algorithm:

Example 5.7. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = \mathfrak{m}^2$ [8, Example 4.3]. From Example 5.5 we gather all the information we need for the input of Algorithm 2:
INPUT: $b_1 = 1, b_2 = y_1, b_3 = y_2$ $\mathbf{k}$-basis of $I^\perp$; $F_1 = y^2, F_2 = y_1y_2, F_3 = y_1^2$ adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$; U_1',U_2' contraction matrices of $\int_\mathfrak{m} I^\perp$.
OUTPUT: $\operatorname{rad}(\mathfrak{a}) = a_2^2 - a_1a_3$.
Then $MGC(A) = \mathbb{P}^2 \backslash \{a_2^2 - a_1a_3 = 0\}$ and any minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ of A is given by a polynomial $F = a_1y_2^2 + a_2y_1y_2 + a_3y_1^2$ such that $a_2^2 - a_1a_3 \neq 0$.

5.3. **Computing** $MGC(A)$ **in colength 2.** Consider a $\mathbf{k}$-basis $b_1, \dots, b_t$ of $I^\perp$ and an adapted $\mathbf{k}$-basis $\overline{F}_1, \dots, \overline{F}_{h_1}, \overline{G}_1, \dots, \overline{G}_{h_2}$ of $\mathcal{L}_{A,2}$ (see Definition 3.5) such that

- $b_1, \dots, b_t, F_1, \dots, F_{h_1}$ is a $\mathbf{k}$-basis of $\int_\mathfrak{m} I^\perp$,
- $b_1, \dots, b_t, F_1, \dots, F_{h_1}, G_1, \dots, G_{h_2}$ is a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^2} I^\perp$.

Throughout this section, we will Consider local Artin rings $A = R/I$ such that $\operatorname{gcl}(A) > 1$. If a minimal Gorenstein cover $G = R/\operatorname{Ann_R} H$ of colength 2 exists, then, by Corollary 3.7, we can assume that H is a polynomial of the form

$$H = \sum_{i=1}^{h_1} \alpha_i F_i + \sum_{i=1}^{h_2} \beta_i G_i, \quad \alpha_i, \beta_i \in \mathbf{k}.$$

We want to obtain conditions on the α's and β's under which H actually generates a minimal Gorenstein cover of colength 2. By definition, $H \in \int_{\mathfrak{m}^2} I^\perp$, hence $x_k \circ H \in \mathfrak{m} \circ \int_\mathfrak{m} \left(\int_\mathfrak{m} I^\perp\right) \subseteq \int_\mathfrak{m} I^\perp$ and

$$x_k \circ H = \sum_{j=1}^{t} \mu_k^j b_j + \sum_{j=1}^{h_1} \rho_k^j F_j, \quad \mu_k^j, \rho_k^j \in \mathbf{k}.$$

Set matrices $A_H = (\mu_k^j)$ and $B_H = (\rho_k^j)$. Let us describe matrix B_H explicitly. We have

$$x_k \circ H = \sum_{i=1}^{h_1} \alpha_i (x_k \circ F_i) + \sum_{i=1}^{h_2} \beta_i (x_k \circ G_i).$$

Note that each $x_k \circ G_i$, for any $1 \leq i \leq h_2$, is in $\int_\mathfrak{m} I^\perp$ and hence it can be decomposed as

$$x_k \circ G_i = \sum_{j=1}^{t} \lambda_j^{k,i} b_j + \sum_{j=1}^{h_1} a_j^{k,i} F_j, \quad \lambda_j^{k,i}, a_j^{k,i} \in \mathbf{k}.$$

Then

$$x_k \circ H = \sum_{i=1}^{h_1} \alpha_i (x_k \circ F_i) + \sum_{i=1}^{h_2} \beta_i \left(\sum_{j=1}^{t} \lambda_j^{k,i} b_j + \sum_{j=1}^{h_1} a_j^{k,i} F_j \right) = b + \sum_{j=1}^{h_1} \left(\sum_{i=1}^{h_2} \beta_i a_j^{k,i} \right) F_j,$$

where $b := \sum_{i=1}^{h_1} \alpha_i (x_k \circ F_i) + \sum_{i=1}^{h_2} \beta_i \left(\sum_{j=1}^{t} \lambda_j^{k,i} b_j \right) \in I^\perp$. Observe that

$$\rho_k^j = \sum_{i=1}^{h_2} a_j^{k,i} \beta_i, \tag{13}$$

hence the entries of matrix B_H can be regarded as polynomials in variables $\beta_1,\dots,\beta_{h_2}$ with coefficients in $\mathbf{k}$.

Lemma 5.8. *Consider the matrix $B_H=(\rho_k^j)$ as previously defined and let $B'_H=(\varrho_k^j)$ be the matrix of the coefficients of $\overline{L_k\circ H}=\sum_{j=1}^{h_1}\varrho_k^j\overline{F}_j\in\mathcal{L}_{A,1}$ where $L_1,\dots,L_n$ are independent linear forms. Then,*

(i) $\operatorname{rk} B_H=\dim_{\mathbf{k}}\left(\dfrac{\mathfrak{m}\circ H+I^\perp}{I^\perp}\right)$,

(ii) $\operatorname{rk} B'_H=\operatorname{rk} B_H$.

Proof. Since $\overline{x_k\circ H}=\sum_{j=1}^{h_1}\rho_k^j\overline{F}_j$ and $\overline{F}_1,\dots,\overline{F}_{h_1}$ is a $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$, then $\operatorname{rk} B_H=\dim_{\mathbf{k}}\langle\overline{x_1\circ H},\dots,\overline{x_n\circ H}\rangle_{\mathbf{k}}$. Note that $\langle\overline{x_1\circ H},\dots,\overline{x_n\circ H}\rangle_{\mathbf{k}}=(\mathfrak{m}\circ H+I^\perp)/I^\perp\subseteq\mathcal{L}_{A,1}$, hence (i) holds.
For (ii) it will be enough to prove that $\langle\overline{x_1\circ H},\dots,\overline{x_n\circ H}\rangle_{\mathbf{k}}=\langle\overline{L_1\circ H},\dots,\overline{L_n\circ H}\rangle_{\mathbf{k}}$. Indeed, since $L_i=\sum_{j=1}^n\lambda_j^i x_j$ for any $1\le i\le n$, then $\overline{L_i\circ H}=\sum_{j=1}^n\lambda_j^i(\overline{x_j\circ H})\in\langle\overline{x_1\circ H},\dots,\overline{x_n\circ H}\rangle_{\mathbf{k}}$. The reverse inclusion comes from the fact that $L_1,\dots,L_n$ are linearly independent and hence $(L_1,\dots,L_n)=\mathfrak{m}$. □

Lemma 5.9. *With the previous notation, consider a polynomial $H\in\int_{\mathfrak{m}^2}I^\perp$ with coefficients $\beta_1,\dots,\beta_{h_2}$ of $G_1,\dots,G_{h_2}$, respectively, and its corresponding matrix B_H. Then the following are equivalent:*

(i) $B_H\neq 0$,

(ii) $\mathfrak{m}\circ H\nsubseteq I^\perp$,

(iii) $(\beta_1,\dots,\beta_{h_2})\neq(0,\dots,0)$.

Proof. (i) implies (ii). If $B_H\neq 0$, by Lemma 5.8, $(\mathfrak{m}\circ H+I^\perp)/I^\perp\neq 0$ and hence $\mathfrak{m}\circ H\nsubseteq I^\perp$.
(ii) implies (iii). If $\mathfrak{m}\circ H\nsubseteq I^\perp$, by definition $H\notin\int_{\mathfrak{m}}I^\perp$ and hence $H\in\int_{\mathfrak{m}^2}I^\perp\setminus\int_{\mathfrak{m}}I^\perp$. Therefore, some β_i must be non-zero.
(iii) implies (i). Since $G_i\in\int_{\mathfrak{m}^2}I^\perp\setminus\int_{\mathfrak{m}}I^\perp$ for any $1\le i\le h_2$ and, by hypothesis, there is some non-zero β_i, we have that $H\in\int_{\mathfrak{m}^2}I^\perp\setminus\int_{\mathfrak{m}}I^\perp$. We claim that $x_k\circ H\in\int_{\mathfrak{m}}I^\perp\setminus I^\perp$ for some $k\in\{1,\dots,n\}$. Suppose the claim is not true. Then $x_k\circ H\in I^\perp$ for any $1\le k\le n$, or equivalently, $\mathfrak{m}\circ H\subseteq I^\perp$ but this is equivalent to $H\in\int_{\mathfrak{m}}I^\perp$, which is a contradiction. Since

$$x_k\circ H=b+\sum_{j=1}^{h_1}\rho_k^jF_j\in\int_{\mathfrak{m}}I^\perp\setminus I^\perp,\quad b\in I^\perp,$$

for some $k\in\{1,\dots,n\}$, then $\rho_k^j\neq 0$, for some $j\in\{1,\dots,h_1\}$. Therefore, $B_H\neq 0$. □

Lemma 5.10. *Consider the previous setting. If $B_H=0$, then either $\operatorname{gcl}(A)=0$ or $\operatorname{gcl}(A)=1$ or $R/\operatorname{Ann_R}H$ is not a cover of A.*

Proof. If $B_H=0$, then $\mathfrak{m}\circ H\subseteq I^\perp$ and hence $\ell(\langle H\rangle)-1\le\ell(I^\perp)$. If $I^\perp\subseteq\langle H\rangle$, then $G=R/\operatorname{Ann_R}H$ is a Gorenstein cover of A such that $\ell(G)-\ell(A)\le 1$. Therefore, either $\operatorname{gcl}(A)\le 1$ or G is not a cover. □

Since we already have techniques to check whether A has colength 0 or 1, we can focus completely on the case $\operatorname{gcl}(A) > 1$. Then, according to Lemma 5.10, if $G = R/\operatorname{Ann}_{\mathrm{R}} H$ is a Gorenstein cover of A, then $B_H \neq 0$.

Proposition 5.11. *Assume that $B_H \neq 0$. Then $\operatorname{rk} B_H = 1$ if and only if $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^{\perp}$ for some independent linear forms $L_1, \dots, L_n$.*

Proof. Since $B_H \neq 0$, there exists k such that $x_k \circ H \notin I^{\perp}$. Without loss of generality, we can assume that $x_n \circ H \notin I^{\perp}$. If $\operatorname{rk} B_H = 1$, then any other row of B_H must be a multiple of row n. Therefore, for any $1 \leq i \leq n-1$, there exists $\lambda_i \in \mathbf{k}$ such that $(x_i - \lambda_i x_n) \circ H \in I^{\perp}$. Take $L_n := x_n$ and $L_i := x_i - \lambda_i x_n$. Then $L_1, \dots, L_n$ are linearly independent and $L_i \circ H \in I^{\perp}$ for any $1 \leq i \leq n-1$. Moreover, $L_n^2 \circ H \in \mathfrak{m}^2 \circ \int_{\mathfrak{m}^2} I^{\perp} \subseteq I^{\perp}$.

Conversely, let $B'_H = (\varrho_k^j)$ be the matrix of the coefficients of $\overline{L_k \circ H} = \sum_{j=1}^{h_1} \varrho_k^j \overline{F_j} \in \mathcal{L}_{A,1}$. By Lemma 5.8, since $B_H \neq 0$, then $B'_H \neq 0$. By hypothesis, $\overline{L_1 \circ H} = \cdots = \overline{L_{n-1} \circ H} = 0$ in $\mathcal{L}_{A,1}$ but, since $B'_H \neq 0$, then $\overline{L_n \circ H} \neq 0$. Then $\operatorname{rk} B'_H = 1$ and hence, again by Lemma 5.8, $\operatorname{rk} B_H = 1$. $\square$

Recall that $\langle H \rangle = \langle \lambda H \rangle$ for any $\lambda \in \mathbf{k}^*$. Therefore, as pointed out in Theorem 4.2, for any $H \neq 0$, a Gorenstein ring $G = R/\operatorname{Ann}_{\mathrm{R}} H$ can be identified with a point $[H] \in \mathbb{P}_{\mathbf{k}}(\mathcal{L}_{A,2})$ by taking coordinates $(\alpha_1 : \cdots : \alpha_{h_1} : \beta_1 : \cdots : \beta_{h_2})$. Observe that $\mathbb{P}_{\mathbf{k}}(\mathcal{L}_{A,2})$ is a projective space over $\mathbf{k}$ of dimension $h_1 + h_2 - 1$, hence we will denote it by $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$.

On the other hand, by Equation (13), any minor of $B_H = (\rho_k^j)$ is a homogeneous polynomial in variables $\beta_1, \dots, \beta_{h_2}$. Therefore, we can consider the homogeneous ideal $\mathfrak{b}$ generated by all order-2-minors of B_H in $\mathbf{k}[\alpha_1, \dots, \alpha_{h_1}, \beta_1, \dots, \beta_{h_2}]$. Hence $\mathbb{V}_+(\mathfrak{b})$ is the projective variety consisting of all points $[H] \in \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$ such that $\operatorname{rk} B_H \leq 1$.

Remark 5.12. In this section we will use the notation $MGC_2(A)$ to denote the set of points $[H] \in \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$ such that $G = R/\operatorname{Ann}_{\mathrm{R}} H$ is a Gorenstein cover of A with $\ell(G) - \ell(A) = 2$. Since we are considering rings such that $\operatorname{gcl}(A) > 1$, we can characterize rings of higher colength than 2 as those such that $MGC_2(A) = \emptyset$. On the other hand, $\operatorname{gcl}(A) = 2$ if and only if $MGC_2(A) \neq \emptyset$, hence in this case $MGC_2(A) = MGC(A)$, see Definition 4.4 and Remark 4.5.

Corollary 5.13. *Let $A = R/I$ be an Artin ring such that $\operatorname{gcl}(A) = 2$. Then*

$$MGC_2(A) \subseteq \mathbb{V}_+(\mathfrak{b}) \subseteq \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}.$$

Proof. By Proposition 3.2.(ii), points $[H] \in MGC_2(A)$ correspond to Gorenstein covers $G = R/\operatorname{Ann}_{\mathrm{R}} H$ of A such that $I^{\perp} = (L_1, \dots, L_{n-1}, L_n^2) \circ H$ for some $L_1, \dots, L_n$. Since $B_H \neq 0$ by Lemma 5.10, then we can apply Proposition 5.11 to deduce that $\operatorname{rk} B_H = 1$. $\square$

Note that the conditions on the rank of B_H do not provide any information about which particular choices of independent linear forms $L_1, \dots, L_n$ satisfy the inclusion $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^{\perp}$. In fact, it will be enough to understand which are the L_n that meet the requirements. To that end, we fix $L_n = v_1 x_1 + \cdots + v_n x_n$, where $v = (v_1, \dots, v_n) \neq 0$. We can choose linear forms $L_i = \lambda_1^i x_1 + \cdots + \lambda_n^i x_n$, where

$\lambda_i = (\lambda_1^i, \ldots, \lambda_n^i) \neq 0$, for $1 \leq i \leq n-1$, such that $L_1, \ldots, L_n$ are linearly independent and $\lambda_i \cdot v = 0$. Observe that the k-vector space generated by $L_1, \ldots, L_{n-1}$ can be expressed in terms of $v_1, \ldots, v_n$, that is,

$$\langle L_1, \ldots, L_{n-1} \rangle_{\mathbf{k}} = \langle v_l x_k - v_k x_l : 1 \leq k < l \leq n \rangle_{\mathbf{k}}.$$

Let us now add the coefficients of L_n to matrix B_H by defining the following matrix depending both on H and v:

$$C_{H,v} := \begin{pmatrix} \rho_1^1 & \ldots & \rho_1^{h_1} & v_1 \\ \vdots & & \vdots & \vdots \\ \rho_n^1 & \ldots & \rho_n^{h_1} & v_n \end{pmatrix}.$$

Proposition 5.14. *Assume $B_H \neq 0$. Consider $L_1, \ldots, L_n$ linearly independent linear forms with $L_n = v_1 x_1 + \cdots + v_n x_n$, $v = (v_1, \ldots, v_n) \neq 0$, and $L_i = \lambda_1^i x_1 + \cdots + \lambda_n^i x_n$, $\lambda_i = (\lambda_1^i, \ldots, \lambda_n^i) \neq 0$, such that $\lambda \cdot v = 0$ for any $1 \leq i \leq n-1$. Then $\operatorname{rk} C_{H,v} = 1$ if and only if $(L_1, \ldots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$.*

Proof. If $\operatorname{rk} C_{H,v} = 1$, then all 2-minors of $C_{H,v}$ vanish and, in particular,

$$v_l \rho_k^j - v_k \rho_l^j = 0 \text{ for any } 1 \leq k < l \leq n \text{ and } 1 \leq j \leq h_1. \tag{14}$$

Recall, from Equation (13), that

$$(v_l x_k - v_k x_l) \circ H = b + \sum_{j=1}^{h_1} \left(v_l \rho_k^j - v_k \rho_l^j \right) F_j, \text{ where } b \in I^\perp, \tag{15}$$

hence $(v_l x_k - v_k x_l) \circ H \in I^\perp$. Therefore, $L_i \circ H \in I^\perp$ for $1 \leq i \leq n-1$. Moreover, $L_n^2 \circ H \in \mathfrak{m}^2 \circ \int_{\mathfrak{m}^2} I^\perp \subseteq I^\perp$.

Conversely, if $(L_1, \ldots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$, then $\operatorname{rk} B_H = 1$ by Proposition 5.11. Hence $\operatorname{rk} C_{H,v} = 1$ if and only if Equation (14) holds. Since $L_i \circ H \in I^\perp$ for any $1 \leq i \leq n-1$, then $(v_l x_k - v_k x_l) \circ H \in I^\perp$ and we deduce from Equation (15) that Equation (14) is indeed satisfied. $\square$

Definition 5.15. *We say that $v = (v_1, \ldots, v_n)$ is an admissible vector of H if $v \neq 0$ and $v_l \rho_k^j - v_k \rho_l^j = 0$ for any $1 \leq k < l \leq n$ and $1 \leq j \leq h_1$.*

Lemma 5.16. *Given a polynomial H of the previous form such that $\operatorname{rk} B_H = 1$:*

(i) *there always exists an admissible vector $v \in \mathbf{k}^n$ of H;*
(ii) *if $w \in \mathbf{k}^n$ such that $w = \lambda v$, with $\lambda \in \mathbf{k}^*$, then w is an admissible vector of H;*
(iii) *the admissible vector of H is unique up to multiplication by elements of $\mathbf{k}^*$.*

Proof. (i) Since $\operatorname{rk}_H B = 1$, Proposition 5.11 ensures the existence of linearly independent linear forms $L_1, \ldots, L_n$ such that $(L_1, \ldots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$. By Proposition 5.14, the vector whose components are the coefficients of L_n is admissible.
(ii) Since v is admissible, $w = \lambda v \neq 0$ and $w_l \rho_k^j - w_k \rho_l^j = \lambda(v_l \rho_k^j - v_k \rho_l^j) = 0$.
(iii) Since $B_H \neq 0$, there exists $\rho_k^j \neq 0$ for some $1 \leq j \leq h_1$ and $1 \leq k \leq n$. We will first

prove that $v_k \neq 0$. Suppose that $v_k = 0$. By Definition 5.15, there exists $v_i \neq 0$, $i \neq k$, and $v_i \rho_k^j - v_k \rho_i^j = 0$. Then $v_i \rho_k^j = 0$ and we reach a contradiction.

Consider now $w = (w_1, \dots, w_n)$ admissible with respect to H. From $\rho_k^j v_l - \rho_l^j v_k = 0$ and $\rho_k^j w_l - \rho_l^j w_k = 0$, we get $v_l = \left(\rho_l^j / \rho_k^j\right) v_k$ and $w_l = \left(\rho_l^j / \rho_k^j\right) w_k$. Set $\lambda_l := \rho_l^j / \rho_k^j$. For any $1 \leq l \leq n$, with $l \neq k$, from $v_l = \lambda_l v_k$ and $w_l = \lambda_l w_k$, we deduce that $w_l = (w_k / v_k) v_l$. Hence $w = \lambda v$, where $\lambda = w_k / v_k$, and any two admissible vectors of H are linearly dependent. □

We now want to provide a geometric interpretation of pairs of polynomials and admissible vectors and describe the variety where they lay. Let us first note that whenever $B_H = 0$, any $v \neq 0$ is an admissible vector. With this observation and Lemma 5.16, for any polynomial H such that $\operatorname{rk} B_H \leq 1$, we can consider its admissible vectors v as points $[v]$ in the projective space $\mathbb{P}_{\mathbf{k}}^{n-1}$ by taking homogeneous coordinates $(v_1 : \cdots : v_n)$.

Let us consider the ideal generated in $\mathbf{k}[\alpha_1, \dots, \alpha_{h_1}, \beta_1, \dots, \beta_{h_2}, v_1, \dots, v_n]$ by polynomials of the form

$$\rho_k^j \rho_m^l - \rho_k^l \rho_m^j, \quad 1 \leq k < m \leq n, 1 \leq j < l \leq h_1 \tag{16}$$

and

$$v_l \rho_k^j - v_k \rho_l^j, \quad 1 \leq k < l \leq n, 1 \leq j \leq h_1. \tag{17}$$

It can be checked that all these polynomials are bihomogeneous polynomials in the sets of variables $\alpha_1, \dots, \alpha_{h_1}, \beta_1, \dots, \beta_{h_2}$ and $v_1, \dots, v_n$. Therefore, this ideal defines a variety in $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$ the points of which satisfy the following equations:

$$\rho_k^j \rho_m^l - \rho_k^l \rho_m^j = 0, \quad 1 \leq k < m \leq n, 1 \leq j < l \leq h_1; \tag{18}$$

$$v_l \rho_k^j - v_k \rho_l^j = 0, \quad 1 \leq k < l \leq n, 1 \leq j \leq h_1. \tag{19}$$

Definition 5.17. *We denote by $\mathfrak{c}$ the ideal in $\mathbf{k}[\alpha_1, \dots, \alpha_{h_1}, \beta_1, \dots, \beta_{h_2}, v_1, \dots, v_n]$ generated by all order 2 minors of $C_{H,v}$. We denote by $\mathbb{V}_+(\mathfrak{c})$ the variety defined by $\mathfrak{c}$ in $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$.*

Lemma 5.18. *With the previous definitions, the set of points of $\mathbb{V}_+(\mathfrak{c})$ is*

$$\left\{([H],[v]) \in \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1} \mid [H] \in \mathbb{V}_+(\mathfrak{b}) \text{ and } v \text{ admissible with respect to } H\right\}.$$

Proof. It follows from Equation (18) and Equation (19). □

Lemma 5.19. *Let π_1 be the projection map from $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1} \longrightarrow \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$. Then $\pi_1(\mathbb{V}_+(\mathfrak{c})) = \mathbb{V}_+(\mathfrak{b})$. Moreover, π_1 is a bijection over the subset of $\mathbb{V}_+(\mathfrak{c})$ where $\operatorname{rk} B_H = 1$.*

Proof. Any element of $\mathbb{V}_+(\mathfrak{c})$ is of the form $([H],[v])$ described in Lemma 5.18. Then $\pi_1([H],[v]) = [H] \in \mathbb{V}_+(\mathfrak{b})$. Conversely, given an element $[H] \in \mathbb{V}_+(\mathfrak{b})$, then $\operatorname{rk} B_H \leq 1$. If $B_H = 0$, then any $v \neq 0$ satisfies $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$. If $\operatorname{rk} B = 1$, by Lemma 5.16, there exist a unique admissible v up to scalar multiplication, hence $([H],[v])$ is the unique point in $\mathbb{V}_+(\mathfrak{c})$ such that $\pi_1([H],[v]) = [H]$. □

From Corollary 5.13, we know that all covers $G = R/\operatorname{Ann}_{\mathrm{R}} H$ of of $A = R/I$ colength 2 correspond to points $[H] \in \mathbb{V}_+(\mathfrak{b})$ but, in general, not all points of $\mathbb{V}_+(\mathfrak{b})$ correspond to such covers. Therefore, we need to identify and remove those $[H]$ such that $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subsetneq I^\perp$.

As $\mathbf{k}$-vector space, $(L_1, \dots, L_{n-1}, L_n^2) \circ H$ is generated by

- $(v_l x_k - v_k x_l) \circ H$, $1 \leq k < l \leq n$;
- $x^\theta \circ H$, $2 \leq |\theta| \leq s+2$.

Since $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$, we can provide an explicit description of these generators with respect to the $\mathbf{k}$-basis $b_1, \dots, b_t$ of $I^\perp$ as follows:

$$(x_k v_l - x_l v_k) \circ H = \sum_{j=1}^{t} \left(v_l \sum_{i=1}^{h_1} \alpha_i \mu_j^{k,i} - v_k \sum_{i=1}^{h_1} \alpha_i \mu_j^{l,i} + v_l \sum_{i=1}^{h_2} \beta_i \lambda_j^{k,i} - v_k \sum_{i=1}^{h_2} \beta_i \lambda_j^{l,i} \right) b_j,$$

for $1 \leq l < k \leq n$, with $x_k \circ F_i = \sum_{j=1}^t \mu_j^{k,i} b_j$ and $x_k \circ G_i = \sum_{j=1}^t \lambda_j^{k,i} b_j + \sum_{j=1}^{h_1} a_j^{k,i} F_j$, $\mu_j^{k,i}, \lambda_j^{k,i}, a_j^{k,i} \in \mathbf{k}$;

$$x^\theta \circ H = \sum_{j=1}^{t} \left(\sum_{i=1}^{h_1} \mu_j^{\theta,i} \alpha_i + \sum_{i=1}^{h_2} \lambda_j^{\theta,i} \beta_i \right) b_j,$$

where $2 \leq |\theta| \leq s+2$, $x^\theta \circ F_i = \sum_{j=1}^t \mu_j^{\theta,i} b_j$ and $x^\theta \circ G_i = \sum_{j=1}^t \lambda_j^{\theta,i} b_j$, $\mu_j^{\theta,i}, \lambda_j^{\theta,i} \in \mathbf{k}$.

We now define matrix $U_{H,v}$ such that its rows are the coefficients of each generator of $(L_1, \dots, L_{n-1}, L_n^2) \circ H$ with respect to the $\mathbf{k}$-basis $b_1, \dots, b_t$ of $I^\perp$:

$$\begin{array}{r|ccc}
 & b_1 & \dots & b_t \\ \hline
(x_2 v_1 - x_1 v_2) \circ H & \varrho_{1,2}^1 & \cdots & \varrho_{1,2}^t \\
\vdots & \vdots & & \vdots \\
(x_n v_{n-1} - x_{n-1} v_n) \circ H & \varrho_{n-1,n}^1 & \cdots & \varrho_{n-1,n}^t \\
x_1^2 \circ H & \varsigma_{(2,0,\dots,0)}^1 & \cdots & \varsigma_{(2,0,\dots,0)}^t \\
x_1 x_2 \circ H & \varsigma_{(1,1,0,\dots,0)}^1 & \cdots & \varsigma_{(1,1,0,\dots,0)}^t \\
\vdots & \vdots & & \vdots \\
x_n^2 \circ H & \varsigma_{(0,\dots,0,2)}^1 & \cdots & \varsigma_{(0,\dots,0,2)}^t \\
\vdots & \vdots & & \vdots \\
x_n^{s+2} \circ H & \varsigma_{(0,\dots,0,s+2)}^1 & \cdots & \varsigma_{(0,\dots,0,s+2)}^t
\end{array}$$

where

$$\varrho_{l,k}^j := v_l \sum_{i=1}^{h_1} \alpha_i \mu_j^{k,i} - v_k \sum_{i=1}^{h_1} \alpha_i \mu_j^{l,i} + v_l \sum_{i=1}^{h_2} \beta_i \lambda_j^{k,i} - v_k \sum_{i=1}^{h_2} \beta_i \lambda_j^{l,i}$$

and

$$\varsigma_\theta^j := \sum_{i=1}^{h_1} \mu_j^{\theta,i} \alpha_i + \sum_{i=1}^{h_2} \lambda_j^{\theta,i} \beta_i.$$

It can be easily checked that the entries of this matrix are either bihomogeneous polynomials $\varrho_{l,k}^j$ in variables $((\alpha, \beta), v)$ of bidegree $(1,1)$ or homogeneous polynomials ς_θ^j in variables (α, β) of degree 1. Let $\mathfrak{a}$ be the ideal in $\mathbf{k}[\alpha_1, \dots, \alpha_{h_1}, \beta_1, \dots, \beta_{h_2}, v_1, \dots, v_n]$

generated by all minors of $U_{H,v}$ of order $t = \dim_{\mathbf{k}} I^{\perp}$. It can be checked that $\mathfrak{a}$ is a bihomogeneous ideal in variables $((\alpha, \beta), v)$, hence we can think of $\mathbb{V}_+(\mathfrak{a})$ as the following variety in $\mathbb{P}^{h_1+h_2-1} \times \mathbb{P}^{n-1}$:

$$\mathbb{V}_+(\mathfrak{a}) = \{([H],[v]) \in \mathbb{P}^{h_1+h_2-1} \times \mathbb{P}^{n-1} \mid \operatorname{rk} U_{H,v} < t\}.$$

Proposition 5.20. *Assume* $\operatorname{gcl}(A) > 1$. *Consider a point* $([H],[v]) \in \mathbb{V}_+(\mathfrak{c}) \subset \mathbb{P}^{h_1+h_2-1} \times \mathbb{P}^{n-1}$. *Then*

$$[H] \in MGC_2(A) \Longleftrightarrow ([H],[v]) \notin \mathbb{V}_+(\mathfrak{a}),$$

Proof. From Corollary 5.13 we deduce that if $[H]$ is a point in $MGC_2(A)$, then $\operatorname{rk} B_H \leq 1$. The same is true for any point $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$. Let us consider these two cases:

Case $B_H = 0$. Since $\operatorname{gcl}(A) > 1$, then $R/\operatorname{Ann}_{\mathrm{R}} H$ is not a Gorenstein cover of A by Lemma 5.10, hence $[H] \notin MGC_2(A)$. On the other hand, as stated in the proof of Lemma 5.19, $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$ for any $v \neq 0$. By Lemma 5.9 and $\operatorname{gcl}(A) \neq 1$, it follows that

$$(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq \mathfrak{m} \circ H \subsetneq I^{\perp}$$

for any $L_1, \dots, L_n$ linearly independent linear forms, where $L_n = v_1 x_1 + \cdots + v_n x_n$. Therefore, the rank of matrix $U_{H,v}$ is always strictly smaller than $\dim_{\mathbf{k}} I^{\perp}$. Hence $([H],[v]) \in \mathbb{V}_+(\mathfrak{a})$ for any $v \neq 0$.

Case $\operatorname{rk} B_H = 1$. If $[H] \in MGC_2(A)$, then there exist $L_1, \dots, L_n$ such that $(L_1, \dots, L_{n-1}, L_n^2) \circ H = I^{\perp}$. Take v as the vector of coefficients of L_n, it is an admissible vector by definition. By Lemma 5.19, $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$ is unique and $\operatorname{rk} U_{H,v} = \dim_{\mathbf{k}} I^{\perp}$. Therefore, $([H],[v]) \notin \mathbb{V}_+(\mathfrak{a})$.

Conversely, if $([H],[v]) \in \mathbb{V}_+(\mathfrak{c}) \cap \mathbb{V}_+(\mathfrak{a})$, then $\operatorname{rk} U_{H,v} < \dim_{\mathbf{k}} I^{\perp}$ and hence $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subsetneq I^{\perp}$, where $L_n = v_1 x_1 + \cdots + v_n x_n$. By unicity of v, no other choice of $L_1, \dots, L_n$ satisfies the inclusion $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subset I^{\perp}$, hence $[H] \notin MGC_2(A)$. □

Corollary 5.21. *Assume* $\operatorname{gcl}(A) > 1$. *With the previous definitions,*

$$MGC_2(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \pi_1\left(\mathbb{V}_+(\mathfrak{c}) \cap \mathbb{V}_+(\mathfrak{a})\right).$$

Proof. It follows from Lemma 5.19 and Proposition 5.20. □

Finally, let us recall the following result for bihomogeneous ideals:

Lemma 5.22. *Let ideals* $\mathfrak{a}, \mathfrak{c}$ *be as previously defined,* $\mathfrak{d} = \mathfrak{a} + \mathfrak{c}$ *the sum ideal and* $\pi_1 : \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1} \longrightarrow \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$ *be the projection map. Let* $\widehat{\mathfrak{d}}$ *be the projective elimination of the ideal* $\mathfrak{d}$ *with respect to variables* $v_1, \dots, v_n$. *Then,*

$$\pi_1(\mathbb{V}_+(\mathfrak{a}) \cap \mathbb{V}_+(\mathfrak{c})) = \mathbb{V}_+(\widehat{\mathfrak{d}}).$$

Proof. See [3, Section 8.5, Exercise 16]. □

We end this section by providing an algorithm to effectively compute the set $MGC_2(A)$ of any ring $A = R/I$ such that $\operatorname{gcl}(A) > 1$.

Algorithm 3 Compute $MGC_2(A)$ of $A = R/I$ with $n \geq 2$ and $\mathrm{gcl}(A) > 1$

Input: s socle degree of $A = R/I$; $b_1, \dots, b_t$ $\mathbf{k}$-basis of the inverse system $I^\perp$; $F_1, \dots, F_{h_1}, G_1, \dots, G_{h_2}$ an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$; $U_1, \dots, U_n$ contraction matrices of $\int_{\mathfrak{m}^2} I^\perp$.

Output: ideals $\mathfrak{b}$ and $\widehat{\mathfrak{d}}$ such that $MGC_2(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}})$.

Steps:

(1) Set $H = \alpha_1 F_1 + \cdots + \alpha_{h_1} F_{h_1} + \beta_1 G_1 + \cdots + \beta_{h_2} G_{h_2}$, where α, β are variables in $\mathbf{k}$. Set column vectors $\mathbf{H} = (0, \dots, 0, \alpha, \beta)^t$ and $v = (v_1, \dots, v_n)^t$ in $R = \mathbf{k}[\alpha, \beta, v]$, where the first t components of $\mathbf{H}$ are zero.

(2) Build matrix $B_H = (\rho_i^j)_{1 \leq i \leq n, 1 \leq j \leq h_1}$, where $U_i \mathbf{H}$ is the column vector $(\mu_i^1, \dots, \mu_i^t, \rho_i^1, \dots, \rho_i^{h_1}, 0, \dots, 0)^t$.

(3) Build matrix $C_{H,v} = \left(\begin{array}{c|c} B_H & v \end{array} \right)$ as an horizontal concatenation of B_H and the column vector v.

(4) Compute the ideal $\mathfrak{c} \subseteq R$ generated by all minors of order 2 of B_H.

(5) Build matrix $U_{H,v}$ as a vertical concatenation of matrices $(\varrho_{l,k}^j)_{1 \leq j \leq h_1, 1 \leq l < k \leq n}$ and $(\varsigma_\theta^j)_{2 \leq |\theta| \leq s+2, 1 \leq j \leq h_1}$, such that $(v_l U_k - v_k U_l)\mathbf{H} = (\varrho_{l,k}^1, \cdots, \varrho_{l,k}^{h_1}, 0, \cdots, 0)^t$ and $U^\theta \mathbf{H} = (\varsigma_\theta^1, \cdots, \varsigma_\theta^{h_1}, 0, \cdots, 0)^t$, with $1 \leq k < l \leq n$ and $2 \leq |\theta| \leq s+2$.

(6) Compute the ideal $\mathfrak{a} \subseteq R$ generated by all minors of order t of $U_{H,v}$ and the ideal $\mathfrak{d} = \mathfrak{a} + \mathfrak{c} \subseteq R$.

(7) Compute $\widehat{\mathfrak{d}} \subseteq R' = \mathbf{k}[\alpha, \beta]$, where $\widehat{\ }$ denotes the projective elimination of the ideal in R with respect to variables $v_1, \dots, v_n$.

(8) Compute the ideal $\mathfrak{b} := \widehat{\mathfrak{c}} \subseteq R'$.

The output of Algorithm 3 can be interpreted as $MGC_2(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}})$. Moreover, any point $[\alpha_1 : \cdots : \alpha_{h_1} : \beta_1 : \cdots : \beta_{h_2}] \in MGC_2(A)$ corresponds to a minimal Gorenstein cover $G = R/\operatorname{Ann_R} H$ of colength 2 of A, where $H = \alpha_1 F_1 + \cdots + \alpha_{h_1} F_{h_1} + \beta_1 G_1 + \cdots + \beta_{h_2} G_{h_2}$. If $MGC_2(A) \neq \emptyset$, then $\mathrm{gcl}(A) = 2$ and hence $MGC(A) = MGC_2(A)$. Otherwise, $\mathrm{gcl}(A) > 2$.

Example 5.23. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = (x_1^2, x_1 x_2^2, x_2^4)$. Applying Algorithm 1 twice we get the necessary input for Algorithm 3:

INPUT: $b_1 = 1, b_2 = y_1, b_3 = y_2, b_4 = y_2^2, b_5 = y_1 y_2, b_6 = y_2^3$ $\mathbf{k}$-basis of $I^\perp$; $F_1 = y_2^4, F_2 = y_1 y_2^2, F_3 = y_1^2, G_1 = y_1^2 y_2, G_2 = y_1 y_2^3, G_3 = y_2^5, G_4 = y_1^3$ adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$; U_1, U_2 contraction matrices of $\int_{\mathfrak{m}^2} I^\perp$.

OUTPUT: $\mathfrak{b} = (b_3 b_4, b_2 b_4)$, $\widehat{\mathfrak{d}} = (b_3 b_4, b_2 b_4, b_2^2 - b_1 b_3)$.

$MGC_2(A) = \mathbb{V}_+(b_3 b_4, b_2 b_4) \backslash \mathbb{V}_+(b_3 b_4, b_2 b_4, b_2^2 - b_1 b_3) = \mathbb{V}_+(b_3 b_4, b_2 b_4) \backslash \mathbb{V}_+(b_2^2 - b_1 b_3)$. Note that if $b_3 b_4 = b_2 b_4 = 0$ and $b_4 \neq 0$, then both b_2 and b_3 are zero and the condition $b_2^2 - b_1 b_3 = 0$ always holds. Therefore, $\mathrm{gcl}(A) = 2$ and hence

$$MGC(A) = \mathbb{V}_+(b_4) \backslash \mathbb{V}_+(b_2^2 - b_1 b_3) \simeq \mathbb{P}^5 \backslash \mathbb{V}_+(b_2^2 - b_1 b_3),$$

where $(a_1 : a_2 : a_3 : b_1 : b_2 : b_3)$ are the coordinates of the points in $\mathbb{P}^5$. Moreover, any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann_R} H$, where

$$H = a_1 y_2^4 + a_2 y_1 y_2^2 + a_3 y_1^2 + b_1 y_1^2 y_2 + b_2 y_1 y_2^3 + b_3 y_2^5$$

satisfies $b_2^2 - b_1 b_3 \neq 0$. All such covers admit (x_1, x_2^2) as the corresponding K_H.

6. Computations

The first aim of this section is to provide a wide range of examples of the computation of the minimal Gorenstein cover variety of a local ring A. In [12], Poonen provides a complete classification of local algebras over an algebraically closed field of length equal or less than 6. Note that, for higher lengths, the number of isomorphism classes is no longer finite. We will go through all algebras of Poonen's list and restrict, for the sake of simplicity, to fields of characteristic zero.

On the other hand, we also intend to test the efficiency of the algorithms by collecting the computation times. We have implemented algorithms 1, 2 and 3 of Section 5 in the commutative algebra software *Singular* [4]. The computer we use runs into the operating system Microsoft Windows 10 Pro and its technical specifications are the following: Surface Pro 3; Processor: 1.90 GHz Intel Core i5-4300U 3 MB SmartCache; Memory: 4GB 1600MHz DDR3.

6.1. **Teter varieties.** In this first part of the section we are interested in the computation of Teter varieties, that is, the $MGC(A)$ variety for local $\mathbf{k}$-algebras A of Gorenstein colength 1. All the results are obtained by running Algorithm 2 in *Singular*.

Example 6.1. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$ and $I = (x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^3, x_3^3)$. Note that $\mathrm{HF}_A = \{1, 3, 2\}$ and $\tau(A) = 3$. The output provided by our implementation of the algorithm in *Singular* [4] is the following:

```
F;
a(4)*x(2)^3+a(1)*x(3)^3+a(6)*x(1)^2+a(5)*x(1)*x(2)+a(3)*x(1)*x(3)+a(2)*x(2)*x(3)
radical(a);
a(1)*a(4)*a(6)
```

We consider points with coordinates $(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in \mathbb{P}^5$. Therefore, $MGC(A) = \mathbb{P}^5 \backslash \mathbb{V}_+(a_1a_4a_6)$ and any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann_R} H$, where $H = a_1y_3^3 + a_2y_2y_3 + a_3y_1y_3 + a_4y_2^3 + a_5y_1y_2 + a_6y_1^2$ with $a_1a_4a_6 \neq 0$.

In Table 1 below we show the computation time (in seconds) of all isomorphism classes of local $\mathbf{k}$-algebras A of $\mathrm{gcl}(A) = 1$ appearing in Poonen's classification [12]. In this table, we list the Hilbert function of $A = R/I$, the expression of the ideal I up to linear isomorphism, the dimension $h-1$ of the projective space $\mathbb{P}^{h-1}$ where the variety $MGC(A)$ lies and the computation time. Note that our implementation of Algorithm 2 includes also the computation of the $\mathbf{k}$-basis of $\int_{\mathfrak{m}} I^{\perp}$, hence the computation time corresponds to the total.

Note that Algorithm 2 also allows us to prove that all the other non-Gorenstein local rings appearing in Poonen's list have Gorenstein colength at least 2.

HF(R/I)	I	h-1	t(s)
$1,2$	$(x_1,x_2)^2$	2	0,06
$1,2,1$	x_1x_2, x_2^2, x_1^3	2	0,06
$1,3$	$(x_1,x_2,x_3)^2$	5	0,13
$1,2,1,1$	x_1^2, x_1x_2, x_2^4	2	0,23
$1,2,2$	x_1x_2, x_1^3, x_2^3	2	0,11
	$x_1x_2^2, x_1^2, x_2^3$	2	0,05
$1,3,1$	$x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2, x_1^3$	5	0,16
$1,4$	$(x_1,x_2,x_3,x_4)^2$	9	2,30
$1,2,1,1,1$	x_1x_2, x_1^5, x_2^2	2	0,17
$1,2,2,1$	x_1x_2, x_1^3, x_2^4	2	0,09
	$x_1^2+x_2^3, x_1x_2^2, x_2^4$	2	0,1
$1,3,1,1$	$x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2, x_1^4$	5	3,05
$1,3,2$	$x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2, x_3^3$	5	0,33
	$x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^3, x_3^3$	5	0,23
$1,4,1$	$x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^2, x_3^2, x_4^2, x_1^3$	9	3,21
$1,5$	$(x_1,x_2,x_3,x_4,x_5)^2$	14	1,25

TABLE 1. Computation times of $MGC(A)$ of local rings $A = R/I$ with $\ell(A) \leq 6$ and $\mathrm{gcl}(A) = 1$.

6.2. **Minimal Gorenstein covers variety in colength 2.** Now we want to compute $MGC(A)$ for $\mathrm{gcl}(A) = 2$. All the examples are obtained by running Algorithm 3 in *Singular*.

Example 6.2. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1,x_2,x_3]\!]$ and $I = (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3)$. Note that $\mathrm{HF}_A = \{1,3,1\}$ and $\tau(A) = 2$. The output provided by our implementation of the algorithm in *Singular* [4] is the following:

```
H;
b(10)*x(1)^3+b(7)*x(1)^2*x(2)+
+b(8)*x(1)*x(2)^2+b(9)*x(2)^3+
+b(1)*x(1)^2*x(3)+b(2)*x(1)*x(2)*x(3)+
+b(3)*x(2)^2*x(3)+b(4)*x(1)*x(3)^2+
+b(6)*x(2)*x(3)^2+b(5)*x(3)^3+
+a(5)*x(1)^2+a(4)*x(1)*x(2)+
+a(3)*x(2)^2+a(2)*x(1)*x(3)+
+a(1)*x(3)^2
radical(b);
_[1]=b(8)^2-b(7)*b(9)
_[2]=b(7)*b(8)-b(9)*b(10)
_[3]=b(6)*b(8)-b(4)*b(9)
_[4]=b(3)*b(8)-b(2)*b(9)
_[5]=b(2)*b(8)-b(1)*b(9)
_[6]=b(1)*b(8)-b(3)*b(10)
_[7]=b(7)^2-b(8)*b(10)
_[8]=b(6)*b(7)-b(4)*b(8)
_[9]=b(4)*b(7)-b(6)*b(10)
_[10]=b(3)*b(7)-b(1)*b(9)
_[11]=b(2)*b(7)-b(3)*b(10)
_[12]=b(1)*b(7)-b(2)*b(10)
_[13]=b(3)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(5)*b(8)
_[15]=b(1)*b(6)-b(5)*b(7)
_[16]=b(2)*b(5)-b(4)*b(6)
_[17]=b(4)^2-b(1)*b(5)
_[18]=b(3)*b(4)-b(5)*b(8)
_[19]=b(2)*b(4)-b(5)*b(7)
_[20]=b(1)*b(4)-b(5)*b(10)
_[21]=b(2)*b(3)-b(4)*b(9)
_[22]=b(1)*b(3)-b(4)*b(8)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(6)*b(10)
_[25]=b(1)^2-b(4)*b(10)
_[26]=b(3)*b(5)*b(10)-b(6)^2*b(10)
_[27]=b(3)^2*b(10)-b(6)*b(9)*b(10)
_[28]=b(4)*b(6)^2-b(5)^2*b(8)
_[29]=b(6)^3*b(10)-b(5)^2*b(9)*b(10)
radical(d);
_[1]=b(8)^2-b(7)*b(9)
_[2]=b(7)*b(8)-b(9)*b(10)
_[3]=b(6)*b(8)-b(4)*b(9)
_[4]=b(3)*b(8)-b(2)*b(9)
_[5]=b(2)*b(8)-b(1)*b(9)
_[6]=b(1)*b(8)-b(3)*b(10)
_[7]=b(7)^2-b(8)*b(10)
_[8]=b(6)*b(7)-b(4)*b(8)
_[9]=b(4)*b(7)-b(6)*b(10)
_[10]=b(3)*b(7)-b(1)*b(9)
_[11]=b(2)*b(7)-b(3)*b(10)
_[12]=b(1)*b(7)-b(2)*b(10)
_[13]=b(3)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(5)*b(8)
```

```
_[15]=b(1)*b(6)-b(5)*b(7)
_[16]=b(2)*b(5)-b(4)*b(6)
_[17]=b(4)^2-b(1)*b(5)
_[18]=b(3)*b(4)-b(5)*b(8)
_[19]=b(2)*b(4)-b(5)*b(7)
_[20]=b(1)*b(4)-b(5)*b(10)
_[21]=b(2)*b(3)-b(4)*b(9)
_[22]=b(1)*b(3)-b(4)*b(8)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(6)*b(10)
_[25]=b(1)^2-b(4)*b(10)
_[26]=b(3)*b(5)*b(10)-b(6)^2*b(10)
_[27]=b(3)^2*b(10)-b(6)*b(9)*b(10)
_[28]=b(4)*b(6)^2-b(5)^2*b(8)
_[29]=a(5)*b(3)*b(5)-a(5)*b(6)^2
_[30]=a(5)*b(3)^2-a(5)*b(6)*b(9)
_[31]=b(6)^3*b(10)-b(5)^2*b(9)*b(10)
_[32]=a(5)*b(6)^3-a(5)*b(5)^2*b(9)
```

We can simplify the output by using the primary decomposition of the ideal $\mathfrak{b} = \bigcap_{i=1}^{k} \mathfrak{b}_i$. Then,

$$MGC(A) = \left(\bigcup_{i=1}^{k} \mathbb{V}_+(\mathfrak{b}_i) \right) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}}) = \bigcup_{i=1}^{k} \left(\mathbb{V}_+(\mathfrak{b}_i) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}}) \right).$$

Singular [4] provides a primary decomposition $\mathfrak{b} = \mathfrak{b}_1 \cap \mathfrak{b}_2$ that satisfies $\mathbb{V}_+(\mathfrak{b}_2) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}}) = \emptyset$. Therefore, we get

$$MGC(A) = \mathbb{V}_+(b_1, b_2, b_4, b_7, b_8, b_{10}, b_3 b_6 - b_5 b_9) \backslash \left(\mathbb{V}_+(a_5) \cup \mathbb{V}_+(-b_6^3 + b_5^2 b_9, b_3 b_5 - b_6^2, b_3^2 - b_6 b_9) \right).$$

in $\mathbb{P}^{14}$. We can eliminate some of the variables and consider $MGC(A)$ to be the following variety:

$$MGC(A) = \mathbb{V}_+(b_3 b_6 - b_5 b_9) \backslash \left(\mathbb{V}_+(a_5) \cup \mathbb{V}_+(b_5^2 b_9 - b_6^3, b_3 b_5 - b_6^2, b_3^2 - b_6 b_9) \right) \subset \mathbb{P}^8.$$

Therefore, any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann_R} H$, where

$$H = a_1 y_3^2 + a_2 y_1 y_3 + a_3 y_2^2 + a_4 y_1 y_2 + a_5 y_1^2 + b_3 y_2^2 y_3 + b_5 y_3^3 + b_6 y_2 y_3^2 + b_9 y_2^3$$

satisfies $b_3 b_6 - b_5 b_9 = 0$, $a_5 \neq 0$ and at least one of the following conditions: $b_5^2 b_9 - b_6^3 \neq 0, b_3 b_5 - b_6^2 \neq 0, b_3^2 - b_6 b_9 \neq 0$.

Moreover, note that $\mathbb{V}_+(\mathfrak{c}) \backslash \mathbb{V}_+(\mathfrak{a}) = \mathbb{V}_+(\mathfrak{c}_1) \backslash \mathbb{V}_+(\mathfrak{a})$, where $\mathfrak{c} = \mathfrak{c}_1 \cap \mathfrak{c}_2$ is the primary decomposition of $\mathfrak{c}$ and $\mathfrak{c}_1 = \mathfrak{b}_1 + (v_1, v_2 b_5 - v_3 b_6, v_2 b_3 - v_3 b_9)$. Hence, any K_H such that $K_H \circ H = I^{\perp}$ will be of the form $K_H = (L_1, L_2, L_3^2)$, where L_1, L_2, L_3 are independent linear forms in R such that $L_3 = v_2 x_2 + v_3 x_3$, with $v_2 b_5 - v_3 b_6 = v_2 b_3 - v_3 b_9 = 0$.

Example 6.3. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$ and $I = (x_1 x_2, x_1 x_3, x_2 x_3, x_2^2, x_3^2 - x_1^3)$. Note that $\mathrm{HF}_A = \{1, 3, 1, 1\}$ and $\tau(A) = 2$. The output provided by our implementation of the algorithm in *Singular* [4] is the following:

```
H;
-b(10)*x(1)^4+b(9)*x(1)^2*x(2)+
+b(7)*x(1)*x(2)^2+b(8)*x(2)^3+
+b(6)*x(1)^2*x(3)+b(1)*x(1)*x(2)*x(3)+
+b(2)*x(2)^2*x(3+b(3)*x(1)*x(3)^2+
+b(4)*x(2)*x(3)^2+b(5)*x(3)^3+
+a(5)*x(1)*x(2)+a(4)*x(2)^2+
+a(3)*x(1)*x(3)+a(2)*x(2)*x(3)+
+a(1)*x(3)^2
radical(b);
_[1]=b(8)*b(10)
_[2]=b(7)*b(10)
_[3]=b(4)*b(10)
_[4]=b(2)*b(10)
_[5]=b(1)*b(10)
_[6]=b(6)*b(8)-b(2)*b(9)
_[7]=b(7)^2-b(8)*b(9)
_[8]=b(6)*b(7)-b(1)*b(9)
_[9]=b(4)*b(7)-b(3)*b(8)
_[10]=b(3)*b(7)-b(4)*b(9)
_[11]=b(2)*b(7)-b(1)*b(8)
_[12]=b(1)*b(7)-b(2)*b(9)
_[13]=b(4)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(4)*b(9)
_[15]=b(1)*b(6)-b(3)*b(9)
_[16]=b(4)^2-b(2)*b(5)
_[17]=b(3)*b(4)-b(1)*b(5)
_[18]=b(2)*b(4)-b(5)*b(8)
_[19]=b(1)*b(4)-b(5)*b(7)
_[20]=b(3)^2-b(5)*b(6)+b(3)*b(10)
_[21]=b(2)*b(3)-b(5)*b(7)
_[22]=b(1)*b(3)-b(5)*b(9)
```

```
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(3)*b(8)
_[25]=b(1)^2-b(4)*b(9)
_[26]=b(5)*b(9)*b(10)
_[27]=b(3)*b(9)*b(10)
radical(d);
_[1]=b(8)*b(10)
_[2]=b(7)*b(10)
_[3]=b(4)*b(10)
_[4]=b(2)*b(10)
_[5]=b(1)*b(10)
_[6]=b(6)*b(8)-b(2)*b(9)
_[7]=b(7)^2-b(8)*b(9)
_[8]=b(6)*b(7)-b(1)*b(9)
_[9]=b(4)*b(7)-b(3)*b(8)
_[10]=b(3)*b(7)-b(4)*b(9)
_[11]=b(2)*b(7)-b(1)*b(8)
_[12]=b(1)*b(7)-b(2)*b(9)
_[13]=b(4)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(4)*b(9)
_[15]=b(1)*b(6)-b(3)*b(9)
_[16]=b(4)^2-b(2)*b(5)
_[17]=b(3)*b(4)-b(1)*b(5)
_[18]=b(2)*b(4)-b(5)*b(8)
_[19]=b(1)*b(4)-b(5)*b(7)
_[20]=b(3)^2-b(5)*b(6)+b(3)*b(10)
_[21]=b(2)*b(3)-b(5)*b(7)
_[22]=b(1)*b(3)-b(5)*b(9)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(3)*b(8)
_[25]=b(1)^2-b(4)*b(9)
_[26]=b(5)*b(9)*b(10)
_[27]=b(3)*b(9)*b(10)
_[28]=a(4)*b(5)*b(10)
_[29]=a(4)*b(3)*b(10)
```

Singular [4] provides a primary decomposition $\mathfrak{b} = \mathfrak{b}_1 \cap \mathfrak{b}_2 \cap \mathfrak{b}_3$ such that $\mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}}) = \mathbb{V}_+(\mathfrak{b}_2)\backslash\mathbb{V}_+(\widehat{\mathfrak{d}})$. Therefore, we get

$$MGC(A) = \mathbb{V}_+(b_1, b_2, b_4, b_7, b_8, b_9, b_3^2 - b_5b_6 + b_3b_{10})\backslash\left(\mathbb{V}_+(a_4) \cup \mathbb{V}_+(b_{10}) \cup \mathbb{V}_+(b_3, b_5)\right).$$

in $\mathbb{P}^{14}$. We can eliminate some of the variables and consider $MGC(A)$ to be the following variety:

$$MGC(A) = \mathbb{V}_+(b_3^2 - b_5b_6 + b_3b_{10})\backslash\left(\mathbb{V}_+(a_4) \cup \mathbb{V}_+(b_{10}) \cup \mathbb{V}_+(b_3, b_5)\right) \subset \mathbb{P}^8.$$

Therefore, any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann_R} H$, where

$$H = a_1y_3^2 + a_2y_2y_3 + a_3y_1y_3 + a_4y_2^2 + a_5y_1y_2 + b_3y_1y_3^2 + b_5y_3^3 + b_6y_1^2y_3 - b_{10}y_1^4$$

satisfies $b_3^2 - b_5b_6 + b_3b_{10} = 0$, $a_4 \neq 0$, $b_{10} \neq 0$ and either $b_3 \neq 0$ or $b_5 \neq 0$ (or both). Moreover, note that $\mathbb{V}_+(\mathfrak{c})\backslash\mathbb{V}_+(\mathfrak{a}) = \mathbb{V}_+(\mathfrak{c}_2)\backslash\mathbb{V}_+(\mathfrak{a})$, where $\mathfrak{c} = \mathfrak{c}_1 \cap \mathfrak{c}_2 \cap \mathfrak{c}_3$ is the primary decomposition of $\mathfrak{c}$ and $\mathfrak{c}_2 = \mathfrak{b}_2 + (v_2, v_1b_5 - v_3b_3 - v_3b_{10}, v_1b_3 - v_3b_6)$. Hence, any K_H such that $K_H \circ H = I^\perp$ will be of the form $K_H = (L_1, L_2, L_3^2)$, where L_1, L_2, L_3 are independent linear forms in R such that $L_3 = v_1x_1 + v_3x_3$, with $v_1b_5 - v_3b_3 - v_3b_{10} = v_1b_3 - v_3b_6 = 0$.

Example 6.4. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$ and $I = (x_1^2, x_2^2, x_3^2, x_1x_2)$. Note that $\operatorname{HF}_A = \{1, 3, 2\}$ and $\tau(A) = 2$. Doing analogous computations to the previous examples, *Singular* provides the following variety:

$$MGC(A) = \mathbb{P}^7\backslash\mathbb{V}_+(b_2^2 - b_1b_3)$$

The coordinates of points in $MGC(A)$ are of the form $(a_1 : \cdots : a_4 : b_1 : b_2 : b_3 : b_4) \in \mathbb{P}^7$ and they correspond to a polynomial

$$H = b_1y_1^2y_3 + b_2y_1y_2y_3 + b_3y_2^2y_3 + b_4y_3^3 + a_1y_3^2 + a_2y_2^2 + a_3y_1y_2 + a_4y_1^2$$

such that $b_2^2 - b_1b_3 \neq 0$. Any $G = R/\operatorname{Ann_R} H$ is a minimal Gorenstein cover of colength 2 of A and all such covers admit (x_1, x_2, x_3^2) as the corresponding K_H.

Example 6.5. Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3, x_4]\!]$ and $I = (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4)$. Note that $\mathrm{HF}_A = \{1, 4, 1\}$ and $\tau(A) = 3$. Doing analogous computations to the previous examples, *Singular* provides the following variety:

$$MGC(A) = \mathbb{V}_+(b_6b_{10} - b_9b_{16}) \setminus (\mathbb{V}_+(d_1) \cup \mathbb{V}_+(d_2)) \subset \mathbb{P}^{12},$$

where $d_1 = (a_7a_9 - a_8^2)$ and $d_2 = (b_9^2b_{16} - b_{10}^3, b_6b_9 - b_{10}^2, b_6^2 - b_{10}b_{16})$. The coordinates of points in $MGC(A)$ are of the form $(a_1 : \cdots : a_9 : b_6 : b_9 : b_{10} : b_{16}) \in \mathbb{P}^{12}$ and they correspond to a polynomial

$$H = b_{16}y_3^3 + b_6y_3^2y_4 + b_{10}y_3y_4^2 + b_9y_4^3 + a_9y_1^2 + a_8y_1y_2 + a_7y_2^2 +$$
$$+ a_6y_1y_3 + a_5y_2y_3 + a_4y_3^2 + a_3y_1y_4 + a_2y_2y_4 + a_1y_4^2$$

such that $G = R/\operatorname{Ann_R} H$ is a minimal Gorenstein cover of colength 2 of A. Moreover, any K_H such that $K_H \circ H = I^\perp$ will be of the form $K_H = (L_1, L_2, L_3, L_4^2)$, where L_1, L_2, L_3, L_4 are independent linear forms in R such that $L_4 = v_3x_3 + v_4x_4$, with $v_3b_9 - v_4b_{10} = v_3b_6 - v_4b_{16} = 0$.

As in the case of colength 1, we now provide a table for the computation times of $MGC(A)$ of all isomorphism classes of local $\mathbf{k}$-algebras A of length equal or less than 6 such that $\mathrm{gcl}(A) = 2$.

HF(R/I)	I	t(s)
$1,3,1$	$x_1x_2, x_1x_3, x_1^2, x_2^2, x_3^2$	0,42
$1,2,2,1$	$x_1^2, x_1x_2^2, x_2^4$	0,18
$1,3,1,1$	$x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2 - x_1^3$	3,56
$1,3,2$	$x_1x_2, x_2x_3, x_3^2, x_2^2 - x_1x_3, x_1^3$	4,4
	$x_1x_2, x_3^2, x_1x_3 - x_2x_3, x_1^2 + x_2^2 - x_1x_3$	1254,34
	$x_1x_2, x_1x_3, x_2^2, x_3^2, x_1^3$	3,33
	$x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 - x_3^2$	4,61
	$x_1^2, x_1x_2, x_2x_3, x_1x_3 + x_2^2 - x_3^2$	4,09
	$x_1^2, x_1x_2, x_2^2, x_3^2$	0,45
$1,4,1$	$x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4$	242,28

TABLE 2. Computation times of $MGC(A)$ of local rings $A = R/I$ with $\ell(A) \leq 6$ and $\mathrm{gcl}(A) = 2$.

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